1. Prove the following summation identity for integers $n \geq 2$:

$$\sum_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

Solution:

1. Base Case $n = 2$

2. Induction Step
   Induction Hypothesis: Property holds when $n = k - 1$, i.e.,

$$\sum_{i=2}^{k-1} \left(1 - \frac{1}{i^2}\right) = \frac{(k-1)+1}{2(k-1)}$$

   Show: Property holds when $n = k$, i.e.,

$$\sum_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k}$$

$$\begin{aligned}
\sum_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) &= \left(1 - \frac{1}{k^2}\right) + \sum_{i=2}^{k-1} \left(1 - \frac{1}{i^2}\right) \\
&= \left(1 - \frac{1}{k^2}\right) + \left(\frac{(k-1)+1}{2(k-1)}\right) && \triangleright \text{ Using I.H.} \\
&= \left(\frac{k^2}{k^2} - \frac{1}{k^2}\right) + \left(\frac{(k-1)+1}{2(k-1)}\right) \\
&= \left(\frac{k^2-1}{k^2}\right) + \left(\frac{(k-1)+1}{2(k-1)}\right) \\
&= \left(\frac{(k+1)(k-1)}{k^2}\right) + \left(\frac{(k-1)+1}{2(k-1)}\right) \\
&= \left(\frac{((k+1)(k-1))(2(k-1))}{(k^2)(2(k-1))}\right) + \left(\frac{(k)\left(k^2\right)}{(2(k-1))(k^2)}\right) \\
&= \frac{((k+1)(k-1))(2(k-1)) + (k)\left(k^2\right)}{(2(k-1))(k^2)} \\
&= \frac{(k-1)(k+1)(2(k-1)) + (k)\left(k^2\right)}{(2(k-1))(k^2)} \\
&= \frac{(k-1)(k+1)(2(k-1)) + \left(k^3\right)}{(2(k-1))(k^2)} \\
&= \frac{(k-1)(k+1)(2k-2)) + \left(k^3\right)}{(2(k-1))(k^2)} \\
&= \frac{(k-1)(k+1)(2k-2)) + \left(k^3\right)}{(2k-2)(k^2)} \\
&= \frac{(k-1)(k+1)(2k-2)) + \left(k^3\right)}{2k^3 - 2k^2} \\
&= \frac{(k-1)(k+1)(2k-2)) + \left(k^3\right)}{2k(k^2 - k)} \\
&= \frac{(k-1)(k+1)(2k-2) + \left(k^3\right)}{2k(k^2 - k)}
\end{aligned}$$