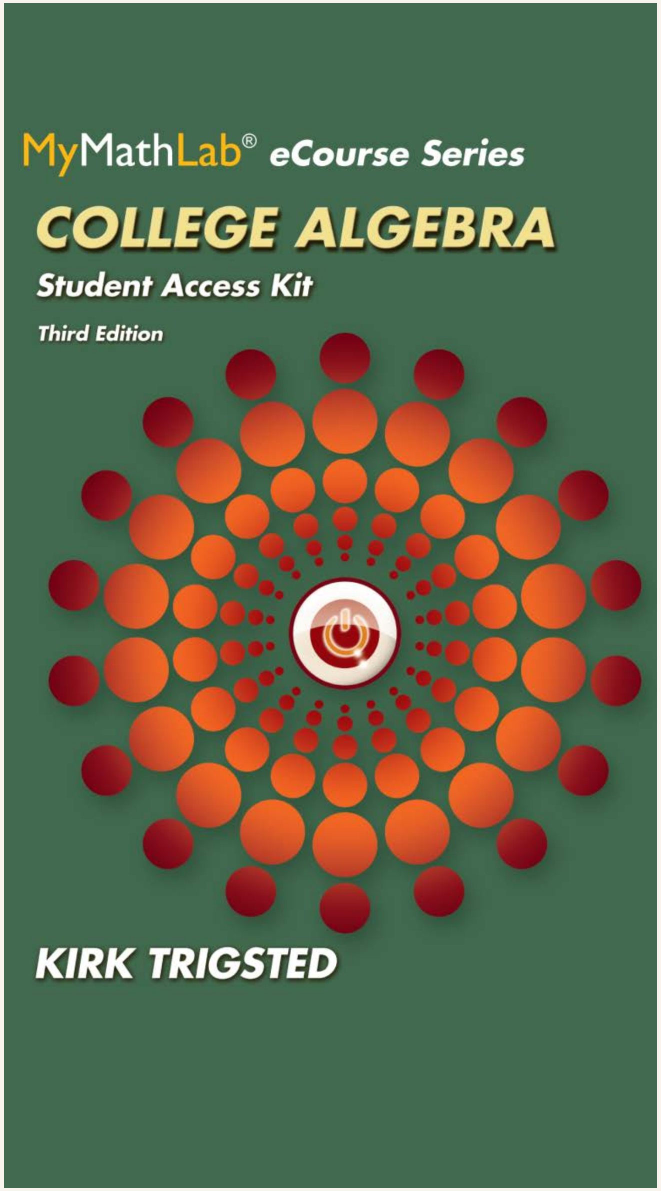
Topic 1.9

Polynomial and Rational Inequalities



OBJECTIVES

1. Solving Polynomial Inequalities
2. Solving Rational Inequalities

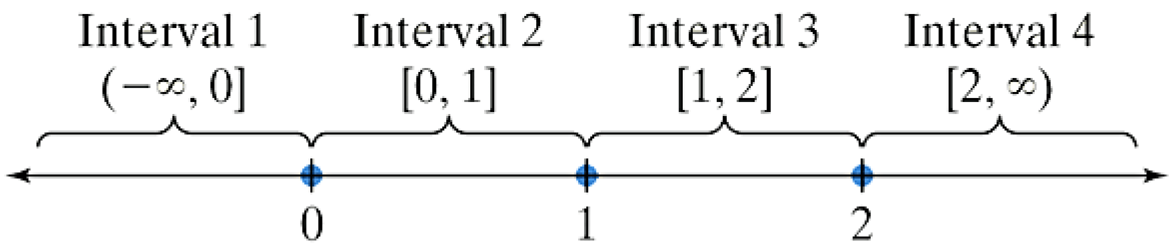
Solving Polynomial Inequalities

EXAMPLE Solve $x^3 - 3x^2 + 2x \geq 0.$

Find boundary points by factoring and setting equal to zero:

$x(x-1)(x-2) = 0$ $\quad\quad$ $x = 0$ or $x - 1 = 0$ or $x - 2 = 0$

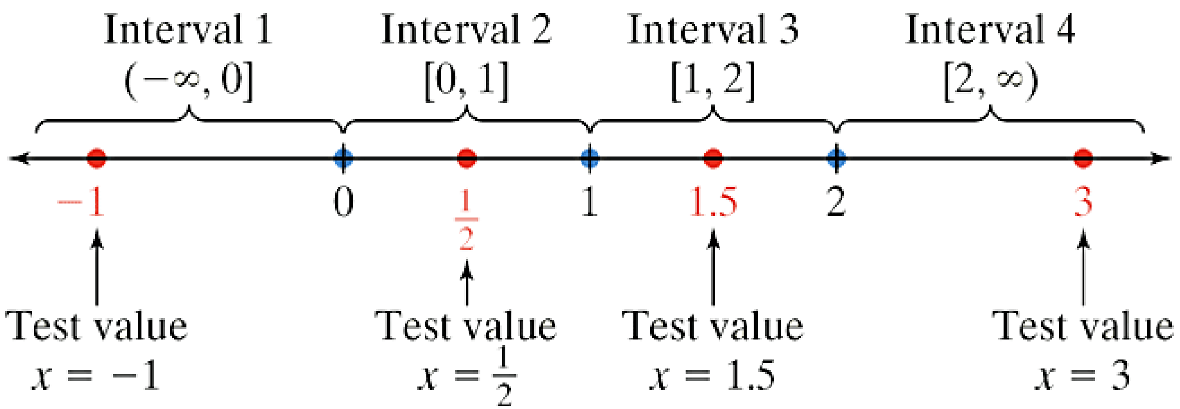
Boundary Points: $x = 0,$ $x = 1$ and $x = 2$



In any of the four intervals formed, the expression must be either *positive* or *negative* throughout the entire interval. To check whether this expression is positive or negative on each interval, pick a number from each interval called a **test value**.

Solving Polynomial Inequalities

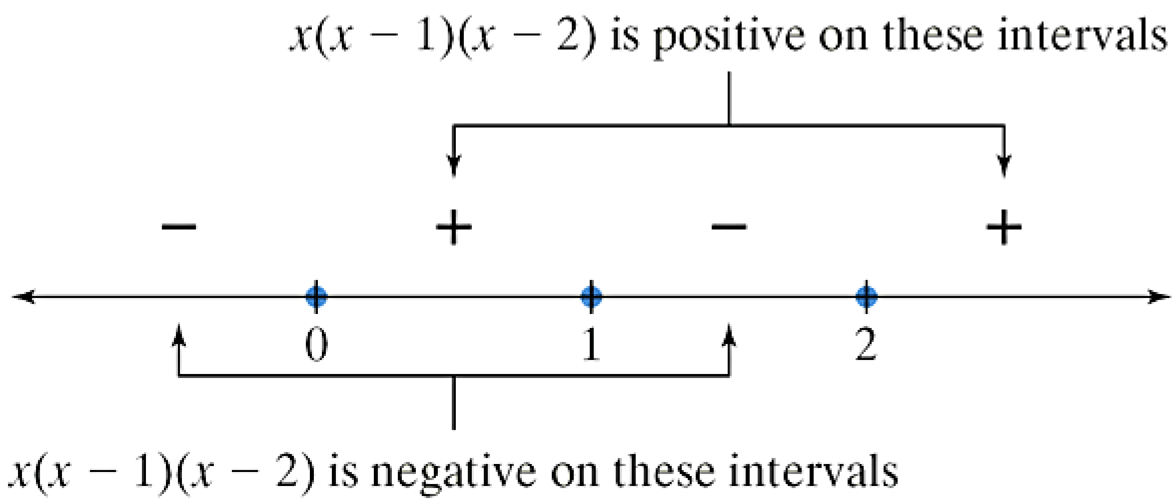
EXAMPLE Solve $x^3 - 3x^2 + 2x \geq 0.$



Interval	Test Value	Substitute Test Value into $x(x-1)(x+2)$	Comment
1. $(-\infty, 0]$	$x = -1$	$(-1)(-1-1)(-1-2) \Rightarrow (-)(-)(-) = -$	Expression is negative on $(-\infty, 0]$
2. $[0, 1]$	$x = \frac{1}{2}$	$\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) \Rightarrow (+)(-)(-) = +$	Expression is positive on $[0, 1]$
3. $[1, 2]$	$x = 1.5$	$(1.5)(1.5-1)(1.5-2) \Rightarrow (+)(+)(-) = -$	Expression is negative on $[1, 2]$
4. $[2, \infty)$	$x = 3$	$(3)(3-1)(3-2) \Rightarrow (+)(+)(+) = +$	Expression is positive on $[2, \infty)$

Solving Polynomial Inequalities

EXAMPLE Solve $x^3 - 3x^2 + 2x \geq 0.$



The expression $x(x-1)(x-2)$ is greater than or equal to zero on the interval: $[0,1]\cup[2,\infty)$.

Solving Polynomial Inequalities

Steps for Solving Polynomial Inequalities

Step 1. Move all terms to one side of the inequality leaving zero on the other side.

Step 2. Factor the nonzero side of the inequality.

Step 3. Find all boundary points by setting the factored polynomial equal to zero.

Step 4. Plot the boundary points on a number line. If the inequality is $\leq$ or $\geq$, then use a solid circle ●. If the inequality is < or >, then use an open circle ○.

Step 5. Now that the number line is divided into intervals, pick a test value from each interval.

Step 6. Substitute the test value into the polynomial, and determine whether the expression is positive or negative on the interval.

Step 7. Determine the intervals that satisfy the inequality.

Solving Polynomial Inequalities

EXAMPLE Solve $x^2 + 5x < 3 - x^2$.

Step 1. Move all terms to one side of the inequality leaving zero on the other side.

$$2x^2 + 5x - 3 < 0$$

Step 2. Factor the nonzero side of the inequality.

$$(2x - 1)(x + 3) < 0$$

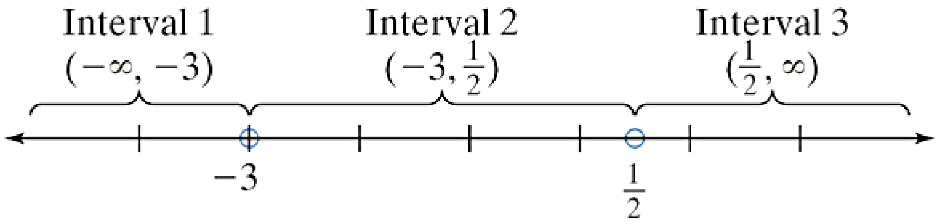
Step 3. Find all boundary points by setting the factored polynomial equal to zero.

The boundary points are $x = -3$ and $x = \frac{1}{2}$.

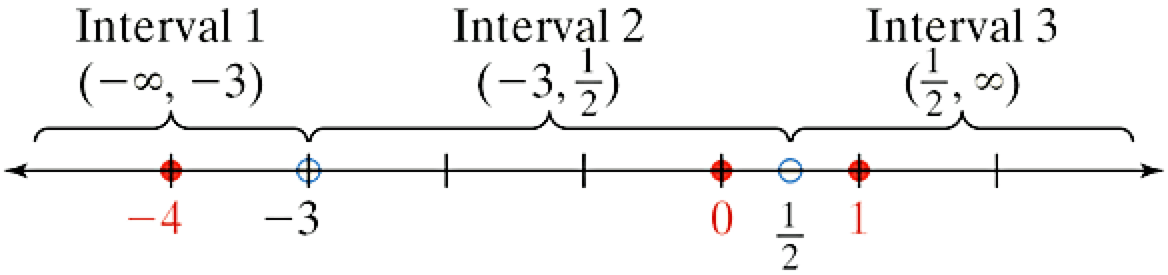
Solving Polynomial Inequalities

EXAMPLE Solve $x^2 + 5x < 3 - x^2$.

Step 4. Plot the boundary points on a number line..



Step 5. Now that the number line is divided into intervals, pick a test value from each interval.



Solving Polynomial Inequalities

EXAMPLE Solve $x^2 + 5x < 3 - x^2.$

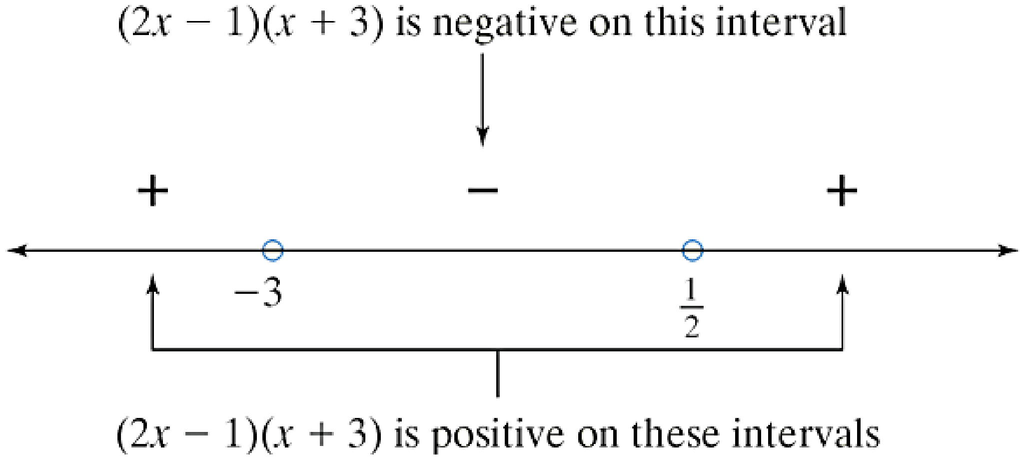
Step 6. Substitute the test value into the polynomial, and determine whether the expression is positive or negative on the interval.

$$x = -4 \; : (2(-4) - 1)((-4) + 3) \Rightarrow (-)(-) = +$$

$$x = 0 \; : (2(0) - 1)((0) + 3) \Rightarrow (-)(+) = -$$

$$x = 1 \; : (2(1) - 1)((1) + 3) \Rightarrow (+)(+) = +$$

Step 7. Determine the intervals that satisfy the inequality.



Because we are looking for values of x that are less than zero (negative values), the solution must be the interval $\left(-3, \frac{1}{2}\right)$.

Solving Rational Inequalities

Steps for Solving Rational Inequalities

Step 1. Move all terms to one side of the inequality leaving zero on the other side.

Step 2. Factor the numerator and denominator of the nonzero side of the inequality and cancel any common factors.

Step 3. Find all boundary points by setting the factored polynomials in the numerator and the denominator equal to zero.

Step 4. Plot the boundary points on a number line.

-For the boundary points found by setting the numerator equal to zero:

If the inequality is $\leq$ or $\geq$, then use a solid circle ●.

If the inequality is $<$ or $>$, then use an open circle ○.

-Use an open circle to represent all boundary points found by setting the denominator equal to zero regardless of the inequality symbol that is used.

Step 5. Now that the number line is divided into intervals, pick a test value from each interval.

Step 6. Substitute the test value into the polynomial, and determine whether the expression is positive or negative on the interval.

Step 7. Determine the intervals that satisfy the inequality.

Solving Rational Inequalities

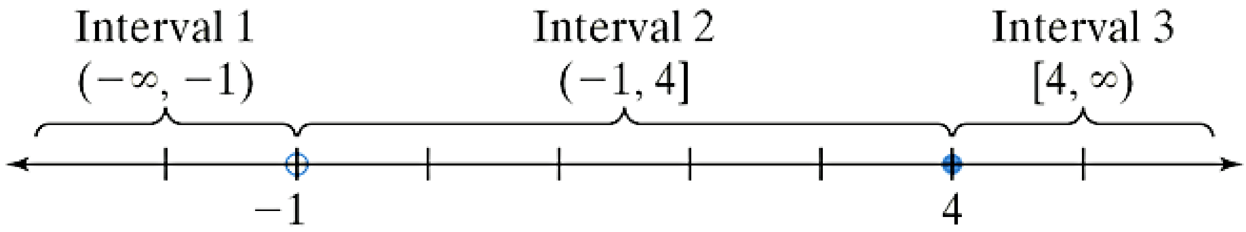
EXAMPLE Solve $\frac{x-4}{x+1} \geq 0.$

Because the inequality is already in completely factored form, we can skip steps 1 and 2 and go right to step 3.

Step 3. Find all boundary points by setting the factored polynomials in the numerator and the denominator equal to zero

Numerator: x - 4 = 0, so x = 4 is a boundary point.
Denominator: x + 1 = 0, so x = -1 is a boundary point.

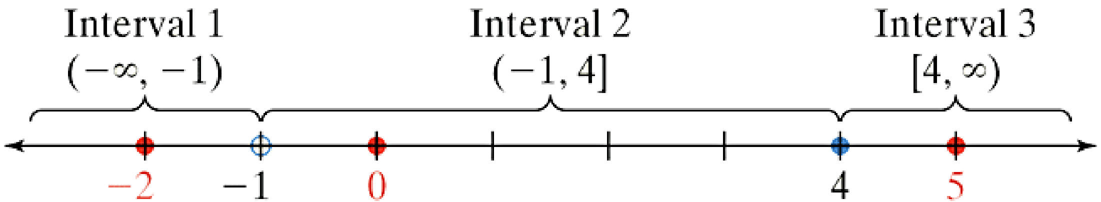
Step 4. Plot the boundary points on a number line.



Solving Rational Inequalities

EXAMPLE Solve $\dfrac{x-4}{x+1} \geq 0.$

Step 5. Now that the number line is divided into intervals, pick a test value from each interval.



Step 6. Substitute the test value into the polynomial, and determine whether the expression is positive or negative on the interval.

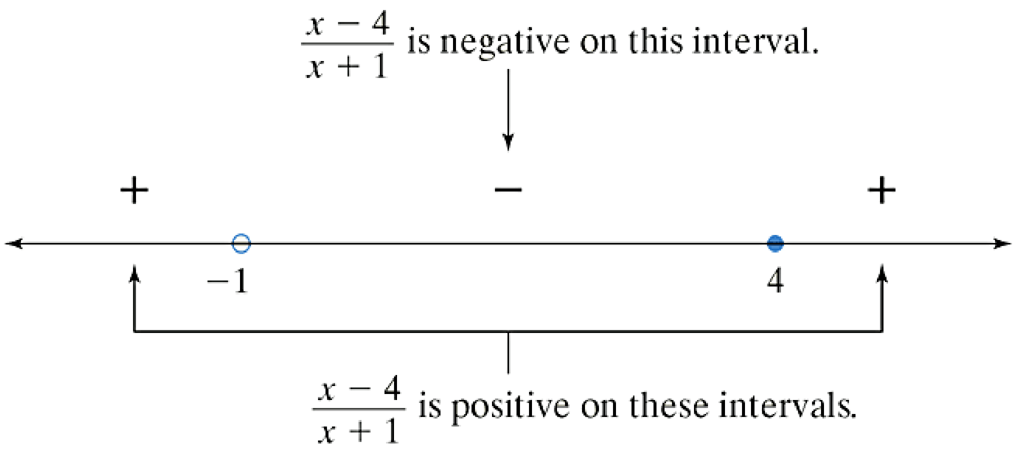
$$x = -2: \quad \frac{(-2-4)}{(-2+1)} \Rightarrow \frac{(-)}{(-)} = +$$

$$x = 0: \quad \frac{(0-4)}{(0+1)} \Rightarrow \frac{(-)}{(+)} = -$$

$$x = 5: \quad \frac{(5-4)}{(5+1)} \Rightarrow \frac{(+)}{(+)} = +$$

Solving Rational Inequalities

EXAMPLE Solve $\dfrac{x-4}{x+1} \geq 0.$



Step 7. Determine the intervals that satisfy the inequality.

Finally, because we are looking for values of x for which the rational expression is greater than or equal to zero, the solution to the inequality is $(-\infty,-1)\cup[4,\infty)$.

Solving Rational Inequalities

CAUTION

You cannot multiply both sides of the inequality by $x+1$ to eliminate the fraction. This is because we do not know whether $x+1$ is negative or positive; therefore, we do not know whether we would need to reverse the direction of the inequality.