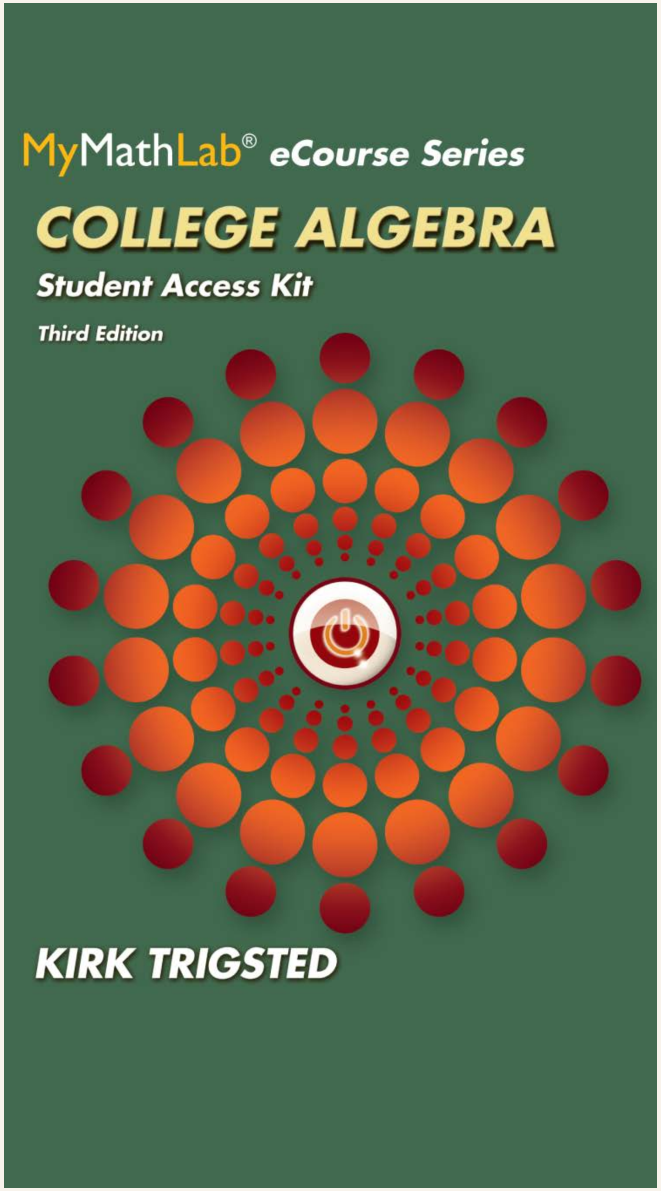
Topic 3.5

The Algebra of Functions; Composite Functions



OBJECTIVES

1. Evaluating a Combined Function
2. Finding the Intersection of Intervals
3. Finding Combined Functions and Their Domains
4. Forming and Evaluating Composite Functions
5. Determining the Domain of Composite Functions

Algebra of Functions

Let f and g be functions, then for all x such that both $f(x)$ and $g(x)$ are defined, the sum, difference, product, and quotient of f and g exist and are defined as follows:

1. The sum of f and g: $(f+g)(x) = f(x) + g(x)$
2. The difference of f and g : $(f-g)(x) = f(x) - g(x)$
3. The product of f and g : $(fg)(x) = f(x)g(x)$
4. The quotient of f and g : $(\frac{f}{g})(x) = \frac{f(x)}{g(x)} \quad \textit{for all } g(x) \neq 0$

Evaluating A Combined Function

EXAMPLE

Let $f(x) = \dfrac{12}{2x+4}$ and $g(x) = \sqrt{x}$. Find each of the following.

a. $(f+g)(1)$

$$(f+g)(1) = f(1) + g(1)$$

$$f(1) = \frac{12}{2(1)+4} = 2$$

$$g(1) = \sqrt{1} = 1$$

$$f(1) + g(1) = 2 + 1$$
$$= 3$$

b. $(f-g)(1)$

$$(f-g)(1) = f(1) - g(1)$$

$$f(1) - g(1) = 2 - 1$$
$$= 1$$

Evaluating A Combined Function

EXAMPLE continued

Let $f(x) = \dfrac{12}{2x+4}$ and $g(x) = \sqrt{x}$. Find each of the following.

c. $(fg)(4)$

$$(fg)(4) = f(4) \cdot g(4)$$

$$f(4) = \frac{12}{2(4)+4} = 1$$

$$g(4) = \sqrt{4} = 2$$

$$f(4) \cdot g(4) = 1 \cdot 2$$
$$= 2$$

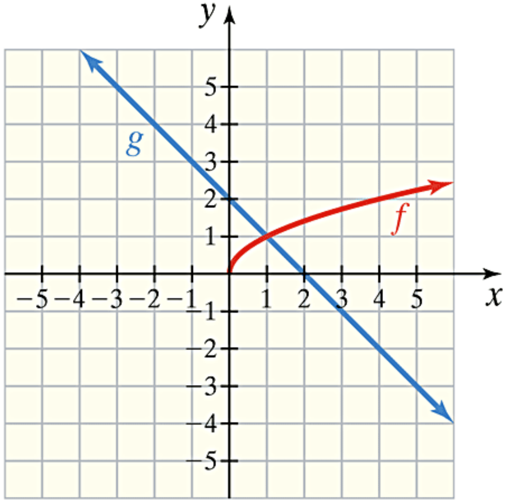
d. $\left(\dfrac{f}{g}\right)(4)$

$$\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)}$$

$$\frac{f(4)}{g(4)} = \frac{1}{2}$$

Evaluate Combined Functions Using a Graph

EXAMPLE Use the graph to evaluate each expression or state that it is undefined.



a. $(f+g)(1)$

$$f(1)+g(1)=1+1$$
$$=2$$

b. $(f-g)(0)$

$$f(0)-g(0)=0-2$$
$$=-2$$

c. $(fg)(4)$

$$f(4)\cdot g(4)=2\cdot -2$$
$$=-4$$

d. $\left(\dfrac{f}{g}\right)(2)$

$$\frac{f(2)}{g(2)}=\frac{f(2)}{0}=undefined$$

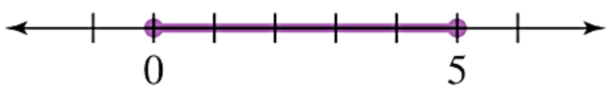
Finding the Intersection of Intervals

EXAMPLE

Find the intersection of the following intervals and graph the set on a number line.

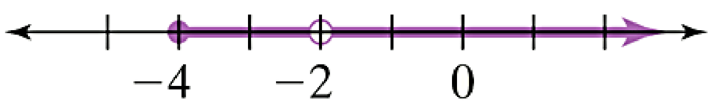
a. $[0,\infty)\cap(-\infty,5]$

$$[0,\infty)\cap(-\infty,5]=[0,5]$$



b. $((-\infty,-2)\cup(-2,\infty))\cap[-4,\infty)$

$$((-\infty,-2)\cup(-2,\infty))\cap[-4,\infty)=[-4,-2)\cup(-2,\infty)$$



Finding Combined Functions and Their Domains

Suppose f is a function with domain A and g is a function with domain B then,

1. The domain of the sum, $f+g$, is the set of all x in $A \cap B$
2. The domain of the difference, $f-g$, is the set of all x in $A \cap B$
3. The domain of the product, fg, is the set of all x in $A \cap B$
4. The domain of the quotient, $\frac{f}{g}$, is the set of all x in $A \cap B$ such that $g(x) \neq 0$

Finding Combined Functions and Their Domains

EXAMPLE Let $f(x)=\frac{x+2}{x-3}$ and $g(x)=\sqrt{4-x}$

Find a. $f+g$, b. $f-g$, c. fg, d. $\frac{f}{g}$, and the domain of each.

Domain of $f(x):(-\infty,3)\cup(3,\infty)$

Domain of $g(x):(-\infty,4]$

$$(f+g)(x)=\frac{x+2}{x-3}+\sqrt{4-x}$$

$$(f-g)(x)=\frac{x+2}{x-3}-\sqrt{4-x}$$

$$(fg)(x)=\frac{(x+2)\sqrt{4-x}}{x-3}$$

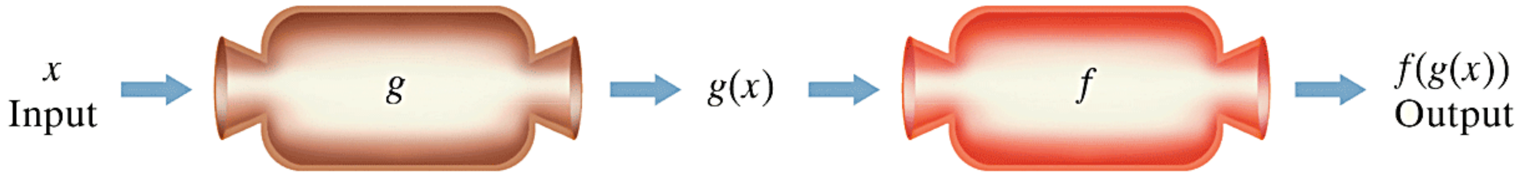
Domain $(-\infty,3)\cup(3,4]$

$$\left(\frac{f}{g}\right)(x)=\frac{\frac{x+2}{x-3}}{\sqrt{4-x}}=\frac{x+2}{(x-3)\sqrt{4-x}}$$

Domain $(-\infty,3)\cup(3,4)$

Composite Function

Given functions *f* and *g*, the composite function, $f \circ g$, (also called the composition of *f* and *g*), is defined by $(f \circ g)(x) = f(g(x))$, provided *g(x)* is in the domain of *f*.



Form and Evaluate Composite Functions

EXAMPLE Let $f(x) = 4x + 1, g(x) = \dfrac{x}{x-2}, h(x) = \sqrt{x+3}$, find

a. $f \circ g$

$$f(g(x))$$

$$f\left(\frac{x}{x-2}\right)$$

$$4\left(\frac{x}{x-2}\right)+1$$

$$\frac{4x}{x-2}+\frac{x-2}{x-2}$$

$$\frac{5x-2}{x-2}$$

b. $g \circ h$

$$g(h(x)) = \frac{\sqrt{x+3}}{\sqrt{x+3}-2}$$

c. $h \circ f \circ g$

$$h(f(g(x))) = \sqrt{\frac{5x-2}{x-2}+3} = \sqrt{\frac{8x-8}{x-2}}$$

Form and Evaluate Composite Functions

EXAMPLE continued

Let $f(x) = 4x + 1, g(x) = \dfrac{x}{x-2}, h(x) = \sqrt{x+3},$ find

d. $(f \circ g)(4),$
or state undefined

$$= \frac{5(4)-2}{(4)-2} = 9$$

e. $(g \circ h)(1),$
or state undefined

$$= \frac{\sqrt{(1)+3}}{\sqrt{(1)+3}-2}$$

$$= \frac{\sqrt{4}}{\sqrt{4}-2}$$

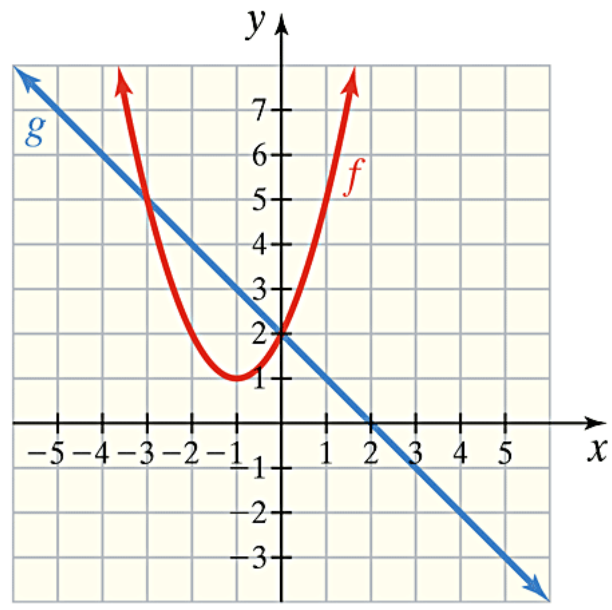
$$= \frac{2}{0} \quad \textit{undefined}$$

f. $(h \circ f \circ g)(6),$
or state undefined

$$= \sqrt{\frac{8(6)-8}{(6)-2}} = \sqrt{10}$$

Evaluate Composite Functions Using a Graph

EXAMPLE



a. $(f \circ g)(4)$ $f(g(4)) = f(-2) = 2$

b. $(g \circ f)(-3)$ $g(f(-3)) = g(5) = -3$

c. $(f \circ f)(-1)$ $f(f(-1)) = f(1) = 5$

d. $(g \circ g)(4)$ $g(g(4)) = g(-2) = 4$

e. $(f \circ g \circ f)(1)$ $f(g(f(1))) = f(g(5)) = f(-3) = 5$

Determining the Domain of Composite Functions

To find the domain of $f \circ g$

Step 1. Find the domain of g

Step 2. Exclude from the domain of g all values of x for which $g(x)$ is not in the domain of f.

Find the Domain of a Composite Function

EXAMPLE

Let $f(x)=\dfrac{-10}{x-4}, g(x)=\sqrt{5-x},$ and $h(x)=\dfrac{x-3}{x+7},$ find the domain:

a. $(f\circ g)(x) = \dfrac{-10}{\sqrt{5-x}-4}$

Domain of $g(x):(-\infty,5]$ (subset of this)

Domain of $f(x):(-\infty,4)\cup(4,\infty)$

$$g(x)=\sqrt{5-x}\neq 4$$

$$5-x\neq 16$$

$$x\neq -11$$

Domain of $(f\circ g)(x):(-\infty,-11)\cup(-11,5]$

b. $(g\circ f)(x) = \sqrt{5-\dfrac{-10}{x-4}} = \sqrt{\dfrac{5x-10}{x-4}}$

Domain of $f(x):(-\infty,4)\cup(4,\infty)$ (subset of this)

Domain of $g(x):(-\infty,5]$

$\dfrac{-10}{x-4}\leq 5$ so, $x\leq 2$ or $x>4$

Domain of $(g\circ f)(x):(-\infty,2]\cup(4,\infty)$