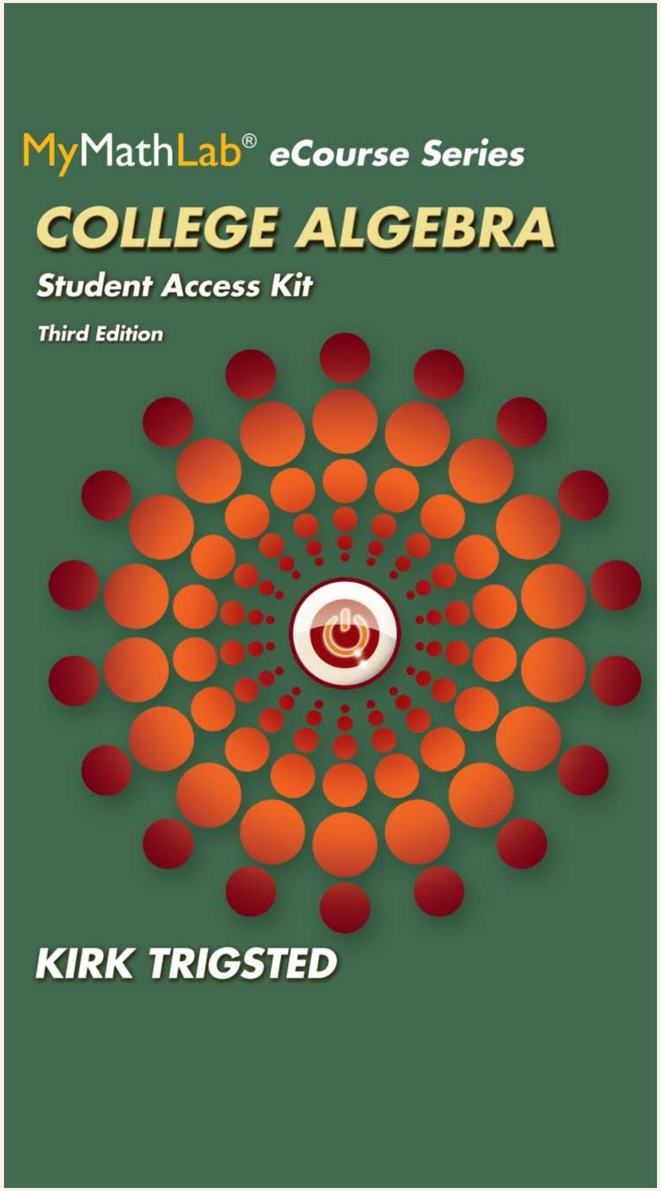
Topic 4.3

The Graphs of Polynomial Functions



OBJECTIVES

1. Understanding the Definition of a Polynomial Function
2. Sketching the Graphs of Power Functions
3. Determining the End Behavior of Polynomial Functions
4. Determining the Intercepts of a Polynomial Function
5. Determining the Real Zeros of Polynomial Functions and Their Multiplicities
6. Sketching the Graph of a Polynomial Function
7. Determining a Possible Equation of a Polynomial Function Giving Its Graph

Polynomial Functions

The function $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x^1 + a_0$ is a polynomial function of degree n, where n is a nonnegative integer. The numbers $a_{1,} a_{2,} \ldots, a_n$ are called the **coefficients** of the polynomial functions. The number a_0 is called the **leading coefficient**, and a_n is called the **constant coefficient**.

Polynomial Functions-cont

- The power of each term must be an integer greater than or equal to zero.
- The coefficients can be real or imaginary.

$$polynomial\ \ functions$$

$$f(x) = \sqrt{5}x^4 - \frac{2}{3}x^2 + 6$$

$$g(x) = \frac{4x^3 - 2x^2 + 5x - 1}{17}$$

$$non - polynomial\ \ function$$

$$g(x) = \frac{4x^3 - 2x^2 + 5x - 1}{x}$$

Understanding the Definition of a Polynomial Function

EXAMPLE

Determine which functions are polynomial functions. If the function is a polynomial function, identify the degree, the leading coefficient, and the constant coefficient.

a. $f(x) = \sqrt{3}x^3 - 2x^2 - \frac{1}{6}$

Polynomial? yes

Degree: 3

Leading Coefficient: $\sqrt{3}$

Constant Coefficient: $-\frac{1}{6}$

b. $g(x) = 4x^5 - 3x^3 + x^2 + \frac{7}{x} - 3$

Polynomial? no

Understanding the Definition of a Polynomial Function

EXAMPLE continued

Determine which functions are polynomial functions. If the function is a polynomial function, identify the degree, the leading coefficient, and the constant coefficient.

c. $h(x) = \dfrac{3x - x^2 + 7x^4}{9} = \dfrac{7}{9}x^4 - \dfrac{1}{9}x^2 + \dfrac{1}{3}x$

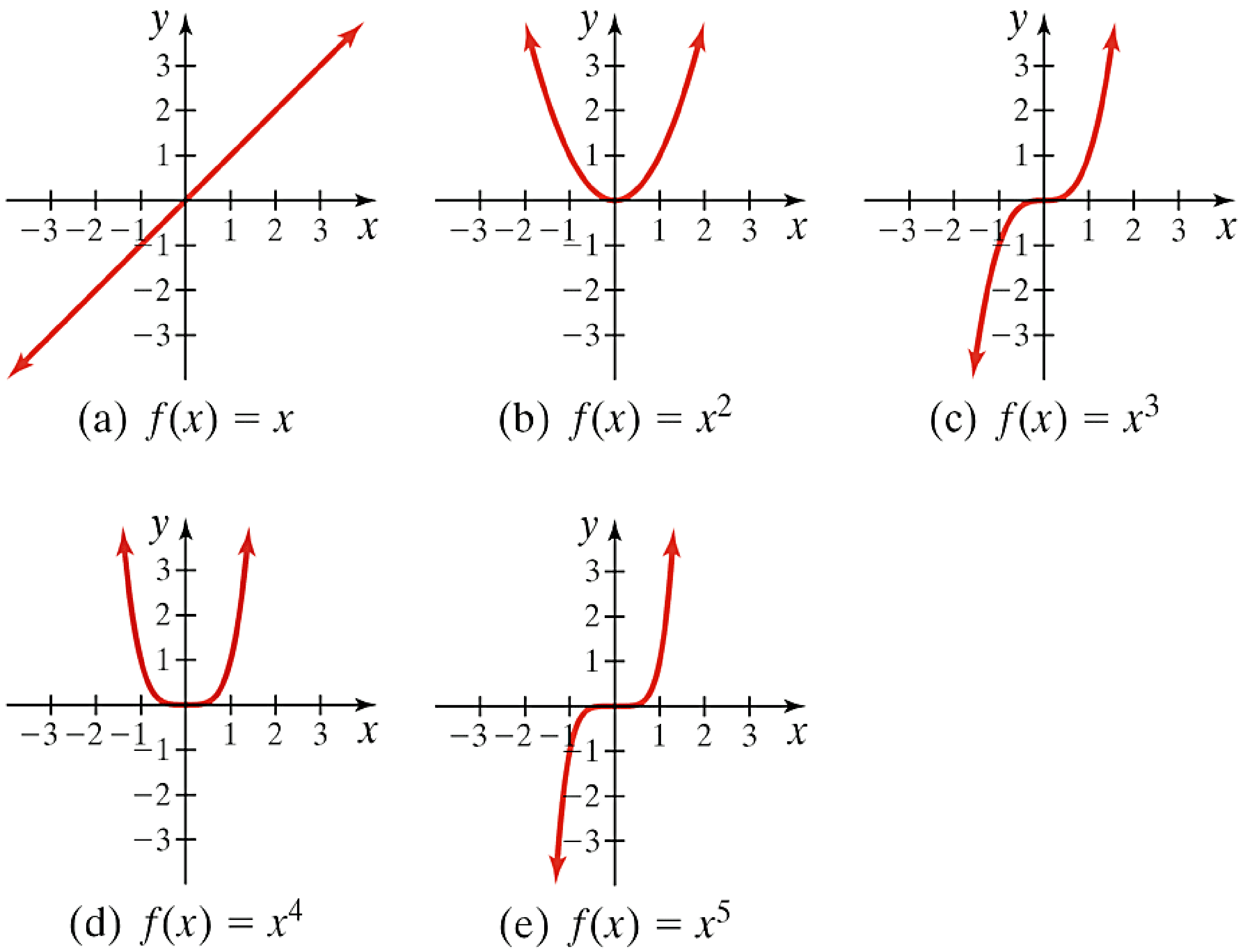
Polynomial? yes

Degree: 4

Leading Coefficient: $\dfrac{7}{9}$

Constant Coefficient: 0

Power functions: $f(x) = ax^n$

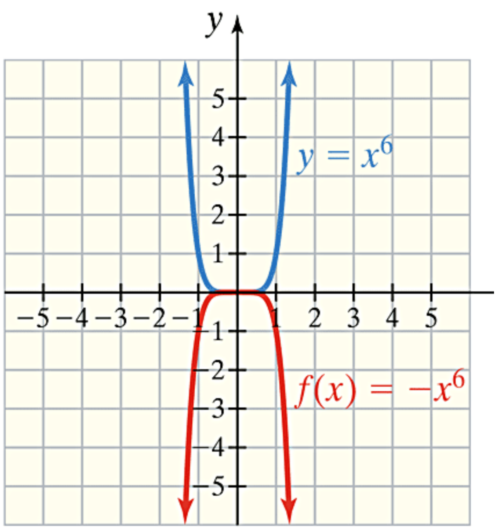


Sketching the Graphs of Power Functions

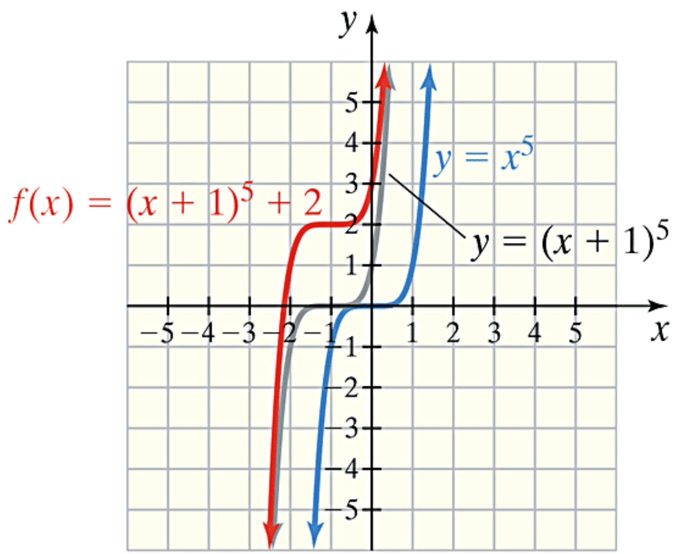
EXAMPLE

Use transformations to sketch the following functions:

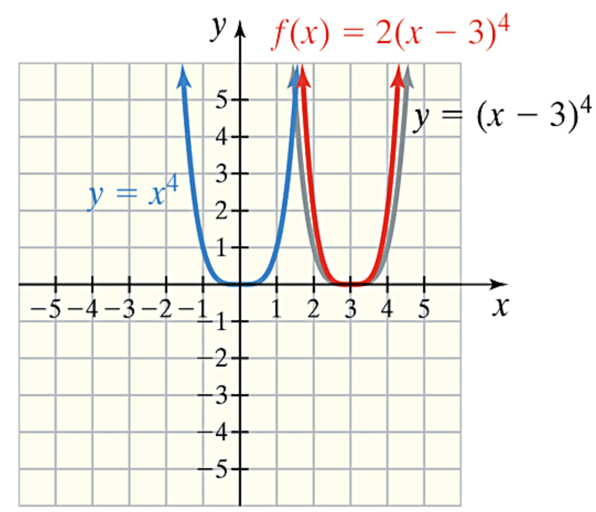
a. $f(x) = -x^6$　　b. $f(x) = (x+1)^5 + 2$　　c. $f(x) = 2(x-3)^4$



(a)



(b)



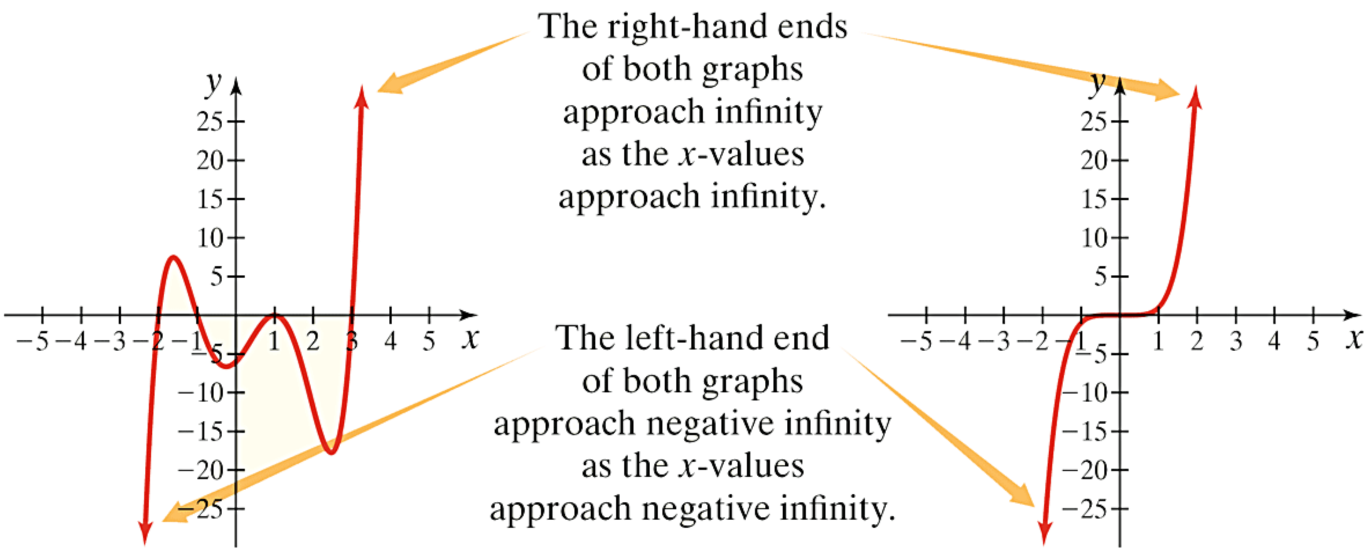
(c)

Determining the End Behavior of Polynomial Functions

End Behavior: The nature of the graph for large values of x in the positive and negative direction.
The end behavior depends upon the leading term

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x^1 + a_0 \quad \text{and} \quad f(x) = a_n x^n$$

have the same end behavior.



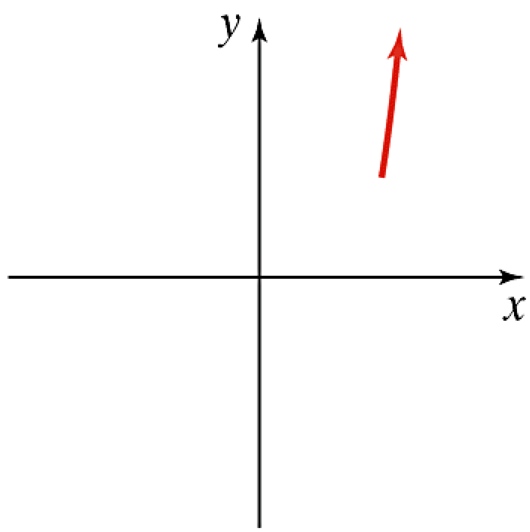
(a) $f(x) = x^5 - 2x^4 - 6x^3 + 8x^2 + 5x - 6$ (b) $y = x^5$

Determining the End Behavior of Polynomial Functions

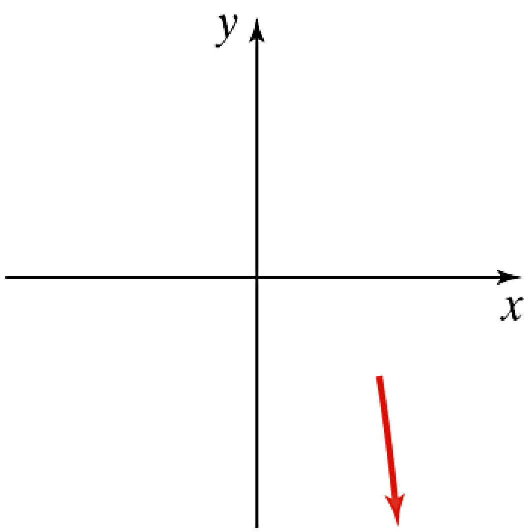
To determine the end behavior of $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$
Look at $f(x) = a_n x^n$
Follow two steps.

Step 1:

If $a_n > 0$, the right-hand behavior "finishes up."



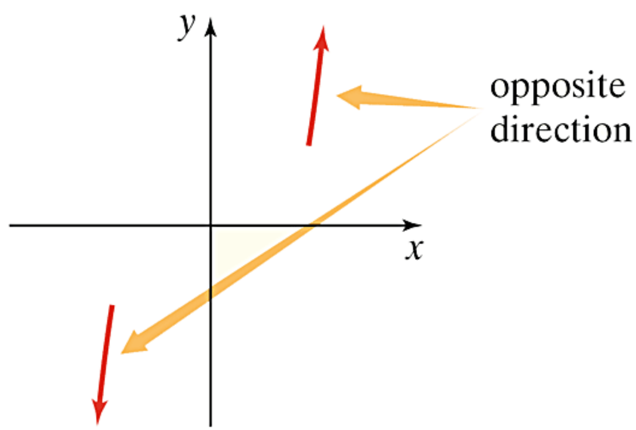
If $a_n < 0$, the right-hand behavior "finishes down."



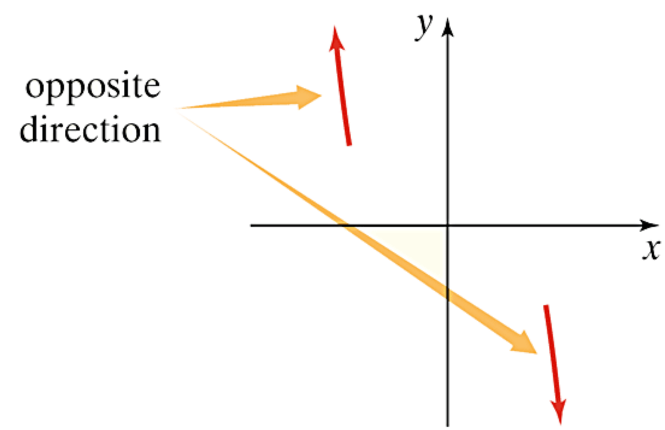
Determining the End Behavior of Polynomial Functions

To determine the end behavior of $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$
Look at $f(x) = a_n x^n$
Follow two steps.

Step 2: If the degree n is odd, the graph has opposite left-hand and right-hand end behavior; that is, the graph "starts" and "finishes" in opposite directions.



$a_n > 0$, odd degree



$a_n < 0$, odd degree

Determining the End Behavior of Polynomial Functions

To determine the end behavior of $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$
Look at $f(x) = a_n x^n$
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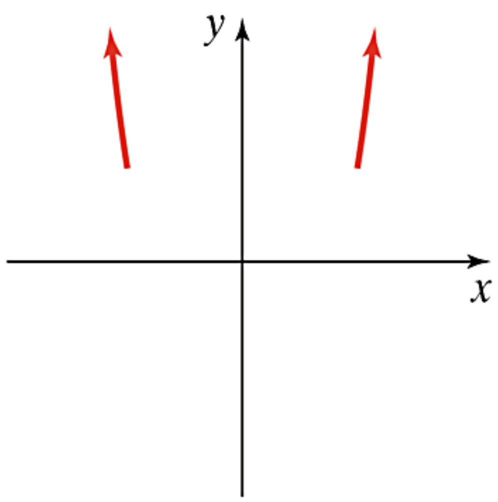
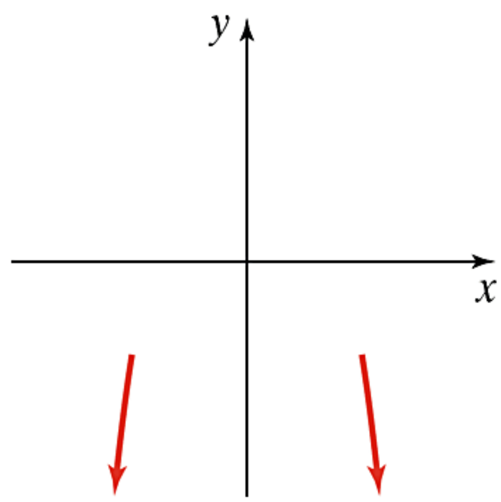


$a_n > 0$, even degree



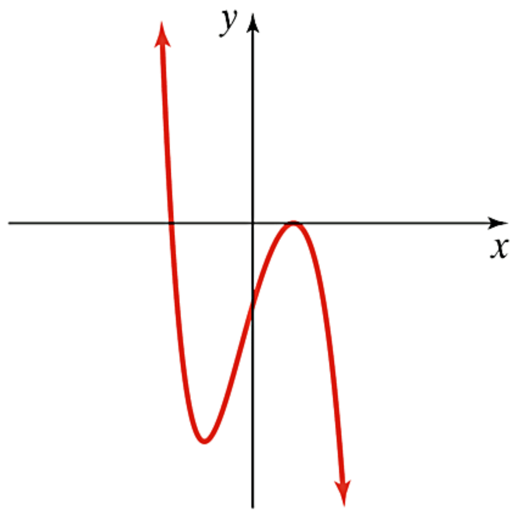
$a_n < 0$, even degree

Determining the End Behavior of Polynomial Functions

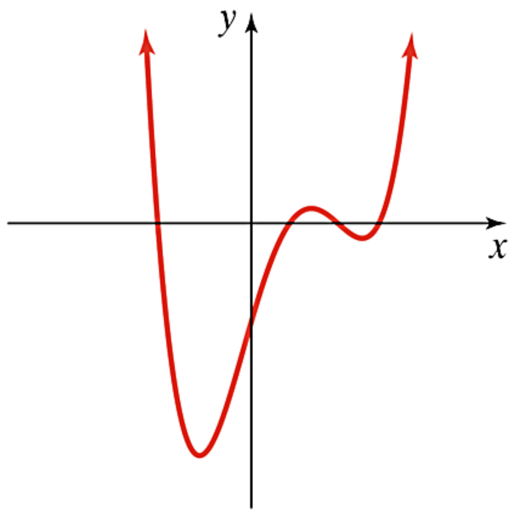
EXAMPLE

Use the end behavior of each graph to determine whether the degree is even or odd and whether the leading coefficient is positive or negative.

a.



b.



Degree: odd

Leading coefficient: negative

Degree: even

Leading coefficient: positive

Determining the intercepts of a Polynomial Function

- *y*-intercept: Found by evaluating $f(0)$
- *x*-intercepts: Found by solving $f(x) = 0$

Determining the intercepts of a Polynomial Function

EXAMPLE

Find the intercepts of the polynomial function $f(x) = x^3 - x^2 - 4x + 4.$

x-intercepts

$$x^3 - x^2 - 4x + 4 = 0$$

$$x^2(x-1) - 4(x-1) = 0$$

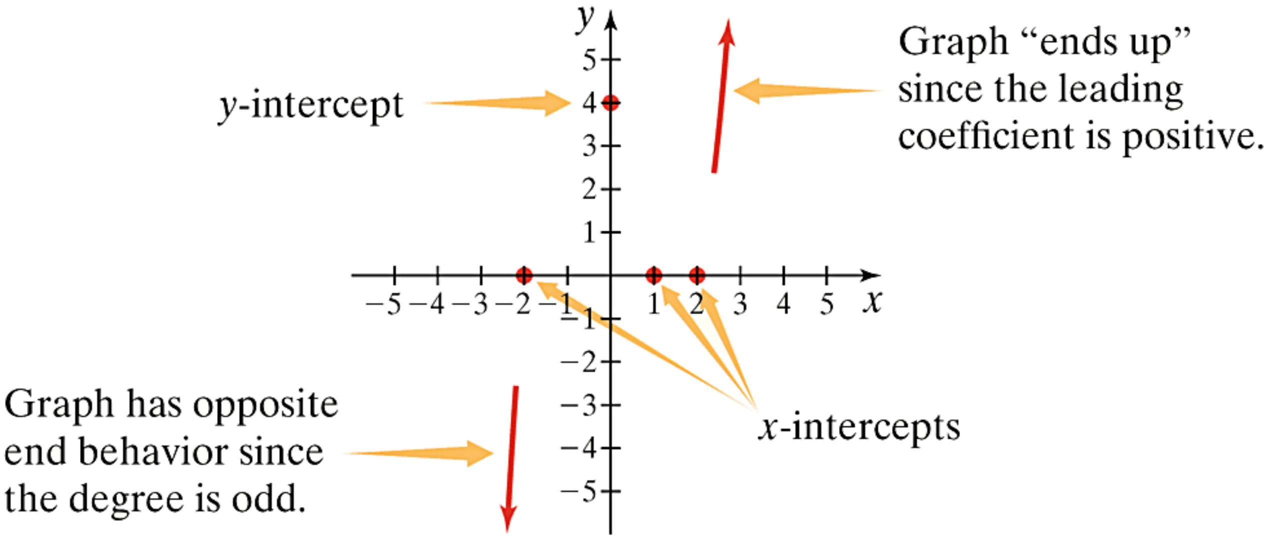
$$(x-1)(x^2-4) = 0$$

$$(x-1)(x-2)(x+2) = 0$$

$$x = 1 \text{ or } x = 2 \text{ or } x = -2$$

y-intercept

$$f(0) = (0)^3 - (0)^2 - 4(0) + 4 = 4$$



Determining the Real Zeros of Polynomial Functions and Their Multiplicities

If $(x - c)$ is a factor of a polynomial function, then $x = c$ is a zero.

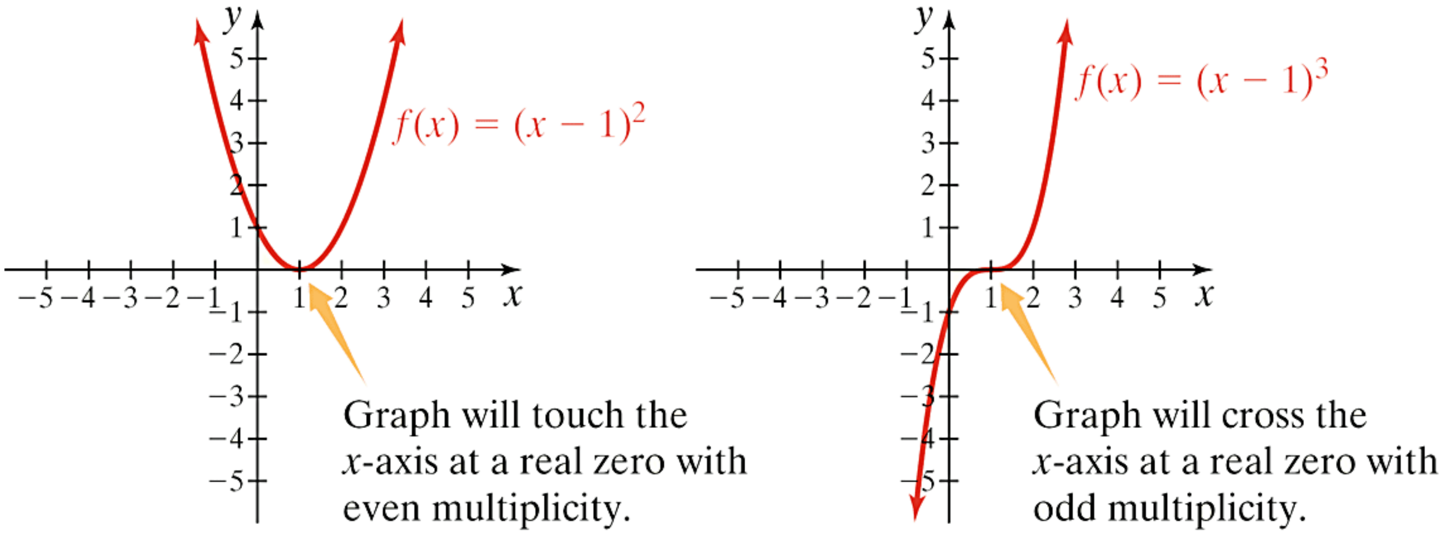
If f is a polynomial function and $x = c$ is a zero, then $(x - c)$ is a factor.

If a factor $(x - c)$ occurs n times in a polynomial, n is an integer greater than 0, the zero $x = c$ is said to have a multiplicity n.

Determining the Real Zeros of Polynomial Functions and Their Multiplicities

In the polynomial $f(x) = (x - 1)^2$, $x = 1$ has a multiplicity of 2.

In the polynomial $f(x) = (x - 1)^3$, $x = 1$ has a multiplicity of 3.

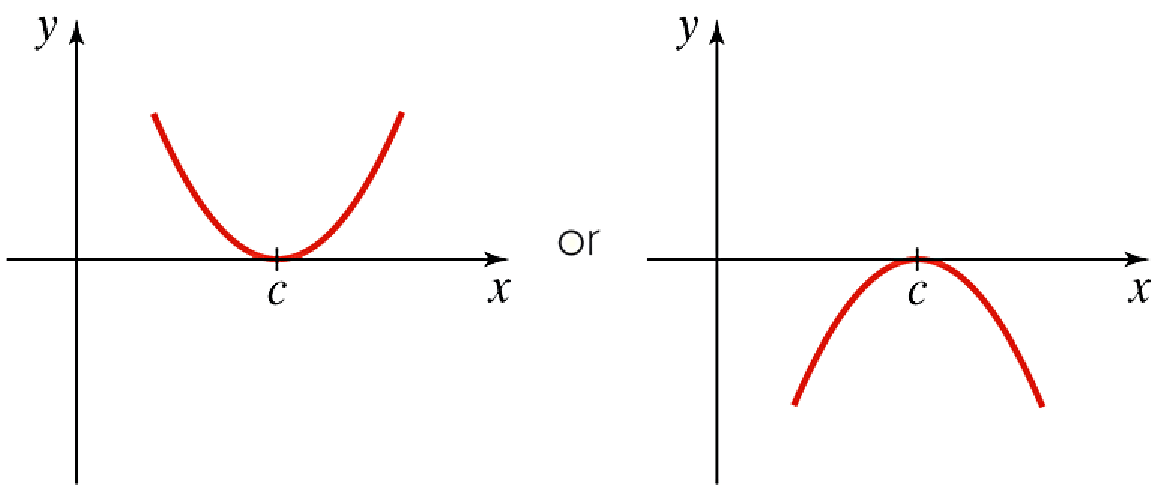


Determining the Real Zeros of Polynomial Functions and Their Multiplicities

Shape of the Graph of a Polynomial Function Near a Zero of Multiplicity *k*

Suppose c is a real zero of a polynomial function f of multiplicity k (where k is a positive integer); that is, $(x - c)^k$ is a factor of f. Then the shape of the graph of f near c is as follows:

If $k > 1$ is even, then the graph touches the x-axis at c.

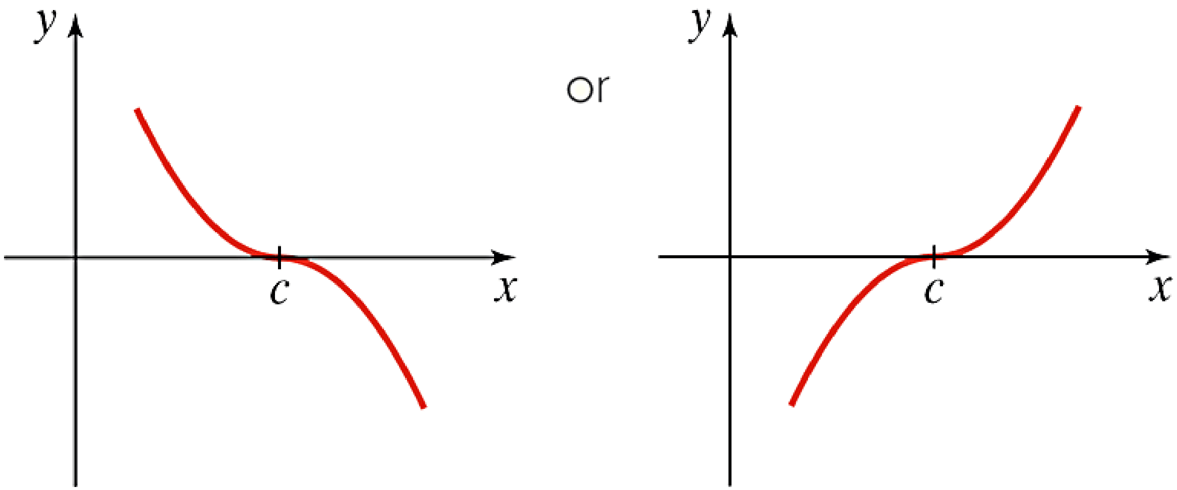


Determining the Real Zeros of Polynomial Functions and Their Multiplicities

Shape of the Graph of a Polynomial Function Near a Zero of Multiplicity *k*

Suppose c is a real zero of a polynomial function f of multiplicity k (where k is a positive integer); that is, $(x - c)^k$ is a factor of f. Then the shape of the graph of f near c is as follows:

If $k \geq 1$ is odd, then the graph crosses the x-axis at c.



Determining the Real Zeros of Polynomial Functions and Their Multiplicities

EXAMPLE

Find all the real zeros, determine the multiplicities of each zero, and decide whether the graph touches or crosses at each zero.

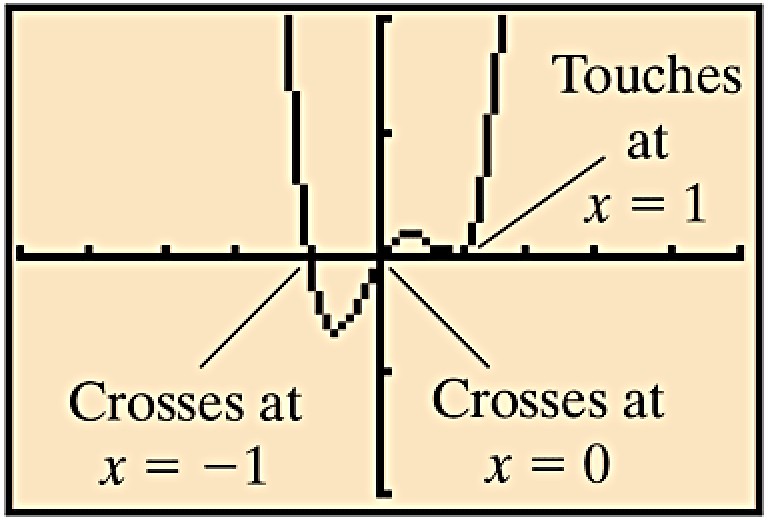
$$f(x) = x(x^2 - 1)(x - 1)$$
$$= x(x - 1)(x + 1)(x - 1)$$
$$= x(x + 1)(x - 1)^2$$

$x = 0$ multiplicity 1

$x = -1$ multiplicity 1

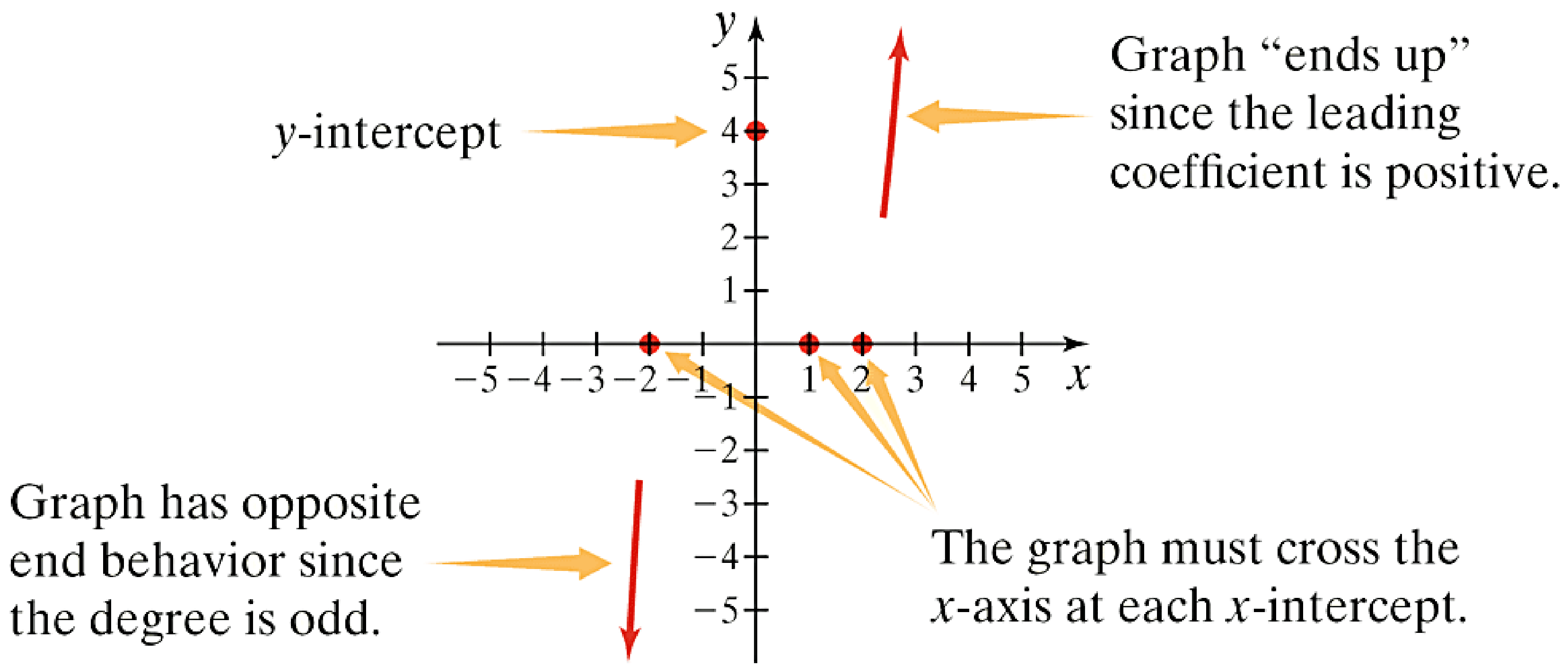
$x = 1$ multiplicity 2

The graph must cross the x-axis at x = 0 and x = -1 and must touch the x-axis at x = 1.



Sketching the Graph of a Polynomial Function

By determining the end behavior of a function and by plotting the x-intercepts and y-intercept, we can begin to sketch the graph.



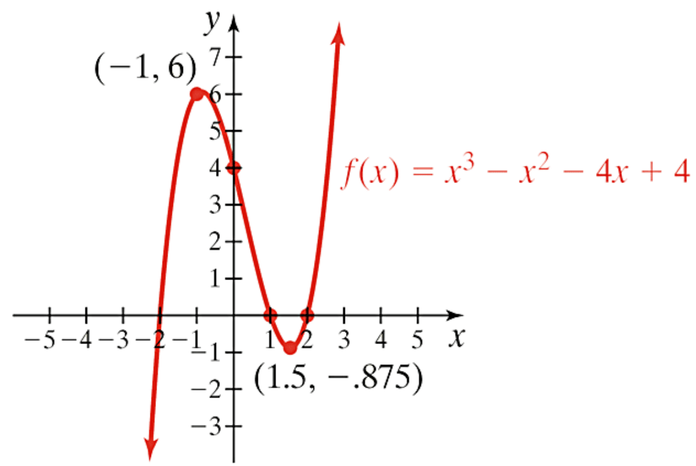
A portion of the graph of $f(x) = x^3 - x^2 - 4x + 4$

Sketching the Graph of a Polynomial Function

To complete the graph, we can create a table of values to plot additional points by choosing a **test value** between each of the zeros. Here, we choose x = -1 and x = 1.5. The approximate graph is sketched.

test values →

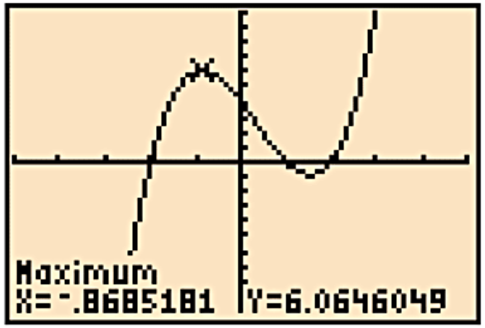
x	$f(x)$
-1	6
1.5	$-.875$

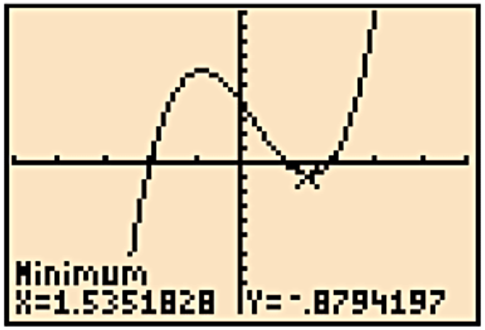


We see from the graph that there are two **turning points**. A polynomial function of degree n has at most n - 1 turning points. A turning point in which the graph changes from increasing to decreasing is also called a **relative maximum**. A turning point in which the graph changes from decreasing to increasing is called a **relative minimum**.

Sketching the Graph of a Polynomial Function

Without the use of calculus (or a graphing utility), there is no way to determine the precise coordinates of the relative minima and relative maxima. Using a graphing utility, we can use the Maximum and Minimum features to determine the coordinates of the relative maxima and minima as follows:





Most graphing calculators have a maximum and minimum feature that approximates the coordinates of a relative maximum or a relative minimum. The graphs above show the approximate coordinates of the relative maximum and relative minimum of $f(x) = x^3 - x^2 - 4x + 4$. Using calculus, it can be shown that the exact coordinates are $\left(\frac{1 - \sqrt{13}}{3}, \frac{70 + 26\sqrt{13}}{27}\right)$ and $\left(\frac{1 + \sqrt{13}}{3}, \frac{70 - 26\sqrt{13}}{27}\right)$, respectively.

Sketching the Graph of a Polynomial Function

Four Step Process for Sketching the Graph of a Polynomial Function

Step 1. Determine the end behavior.

Step 2. Plot the y-intercept $f(0) = a_0$.

Step 3. Completely factor f to find all real zeros and their multiplicities.

Step 4. Choose a test value between each real zero and sketch the graph.

(Remember that without calculus, there is no way to determine the precise coordinates of the turning points.)

Sketching the Graph of a Polynomial Function

EXAMPLE

Use the four-step process to sketch the graphs of the following polynomial functions:

a. $f(x) = -2(x+2)^2(x-1)$

Step 1. Determine the end behavior.

Odd, with leading -, so left hand up, right hand down.

Step 2. Plot the y-intercept $f(0) = a_0$.

$f(0) = -2(0+2)^2(0-1) = 8$

Step 3. Completely factor f to find all real zeros and their multiplicities.

$x = -2$ multiplicity 2

$x = 1$ multiplicity 1

Step 4. Choose a test value between each real zero and sketch the graph.

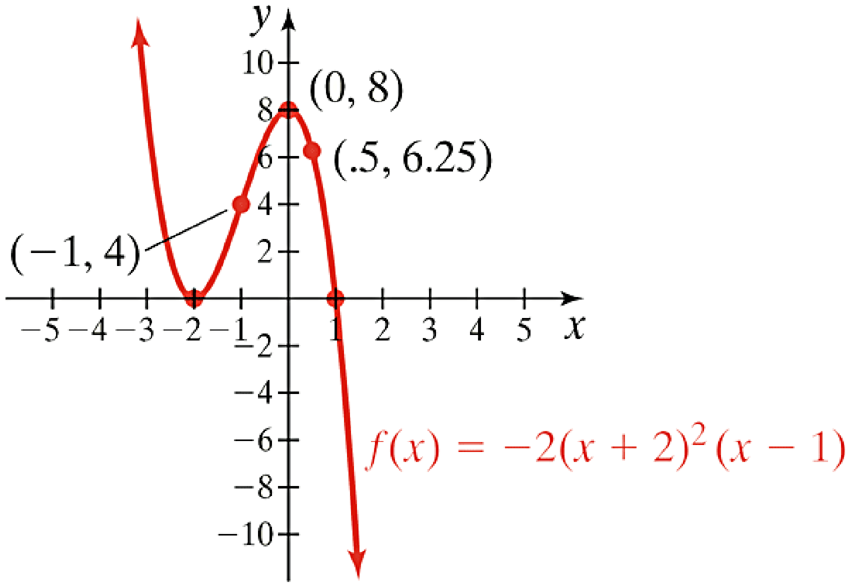
x	y
-1	4
0.5	6.25

Sketching the Graph of a Polynomial Function

EXAMPLE

Use the four-step process to sketch the graphs of the following polynomial functions:

a. $f(x) = -2(x+2)^2(x-1)$



Sketching the Graph of a Polynomial Function

EXAMPLE

Use the four-step process to sketch the graphs of the following polynomial functions:

b. $f(x) = x^4 - 2x^3 - 3x^2$

Step 1. Determine the end behavior. Even, with leading +, so both ends up

Step 2. Plot the y-intercept $f(0) = a_0$. $f(0) = (0)^4 - 2(0)^3 - 3(0)^2 = 0$

Step 3. Completely factor f to find all real zeros and their multiplicities.

$$= x^2\left(x^2 - 2x - 3\right)$$
$$= x^2(x-3)(x+1)$$

$x = 0$ multiplicity 2

$x = 3$ multiplicity 1

$x = -1$ multiplicity 1

Step 4. Choose a test value between each real zero and sketch the graph.

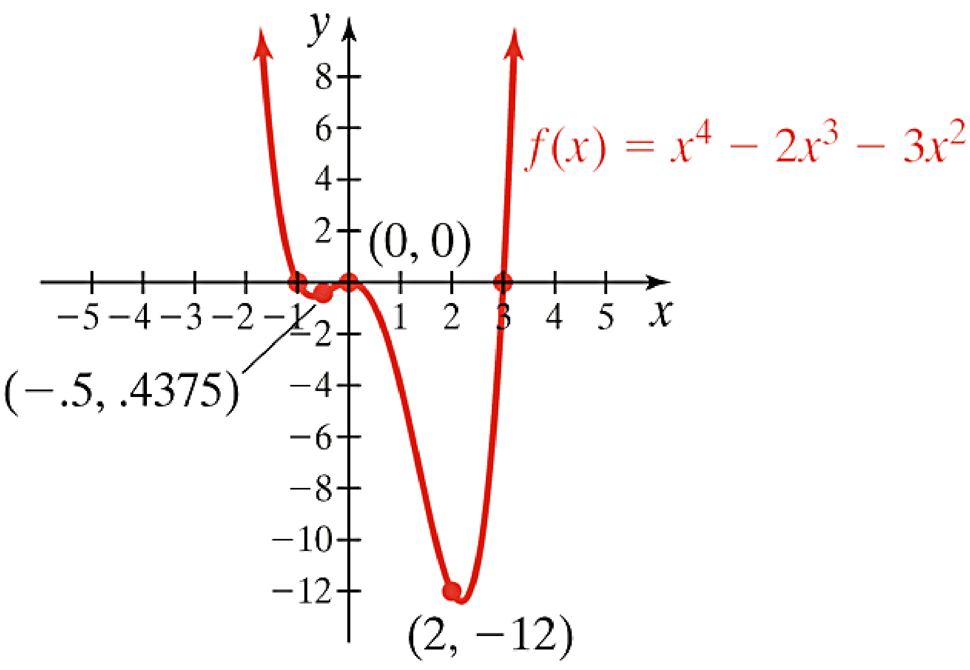
x	y
–0.5	0.4374
2	–12

Sketching the Graph of a Polynomial Function

EXAMPLE

Use the four-step process to sketch the graphs of the following polynomial functions:

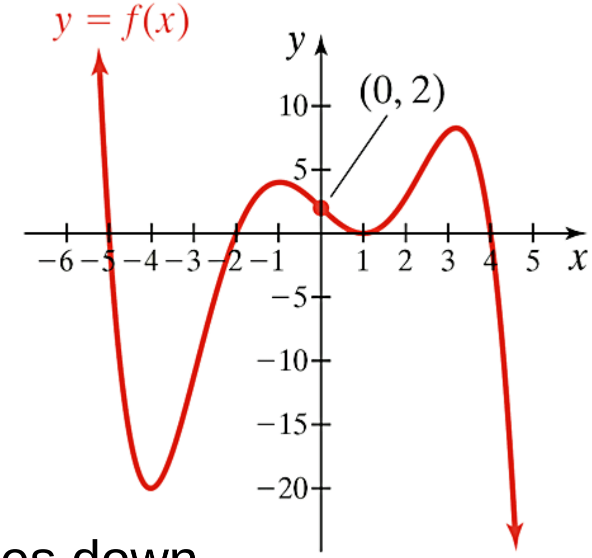
b. $f(x) = x^4 - 2x^3 - 3x^2$



Determining a Possible Equation of a Polynomial Function Given its Graph

EXAMPLE

Analyze the graph to address the following about the polynomial function it represents



a. Is the degree of the polynomial function even or odd?

Odd because the graph starts up and finishes down.

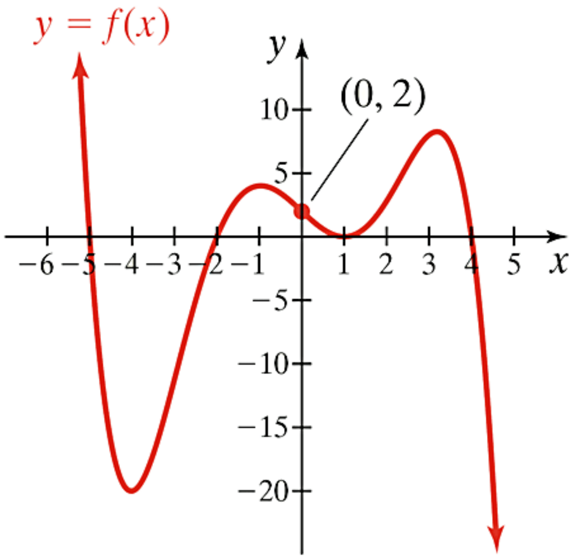
b. Is the leading coefficient positive or negative?

Negative because the right hand behavior finishes down

Determining a Possible Equation of a Polynomial Function Given its Graph

EXAMPLE continued

Analyze the graph to address the following about the polynomial function it represents



c. What is the value of the constant coefficient?

2 because the y-intercept is 2.

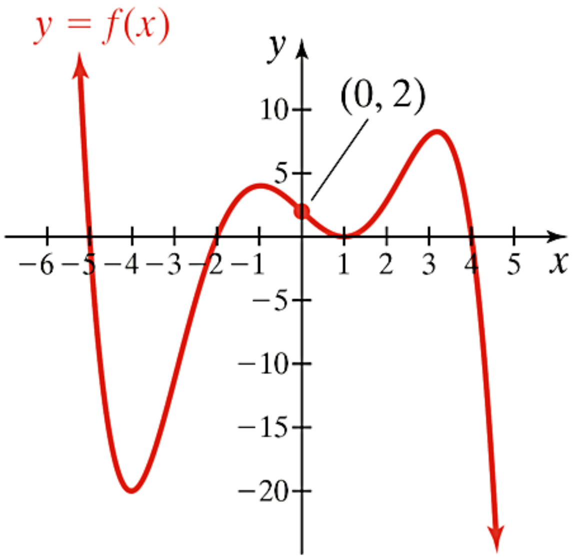
d. Identify the real zeros, and state the multiplicity

x =-5 multiplicity odd
x =-2 multiplicity odd
x =1 multiplicity even
x =4 multiplicity odd

Determining a Possible Equation of a Polynomial Function Given its Graph

EXAMPLE continued

Analyze the graph to address the following about the polynomial function it represents



e. Select from this list a possible function that could be represented by this graph.

i. $f(x) = -\dfrac{1}{20}(x+5)(x+2)(x-1)^2(x-4)$

ii. $f(x) = -\dfrac{1}{800}(x+5)^2(x+2)^2(x-1)(x-4)^2$

iii. $f(x) = \dfrac{1}{20}(x+5)(x+2)(x-1)^2(x-4)$

iv. $f(x) = -\dfrac{1}{10}(x+5)(x+2)(x-1)^2(x-4)$

