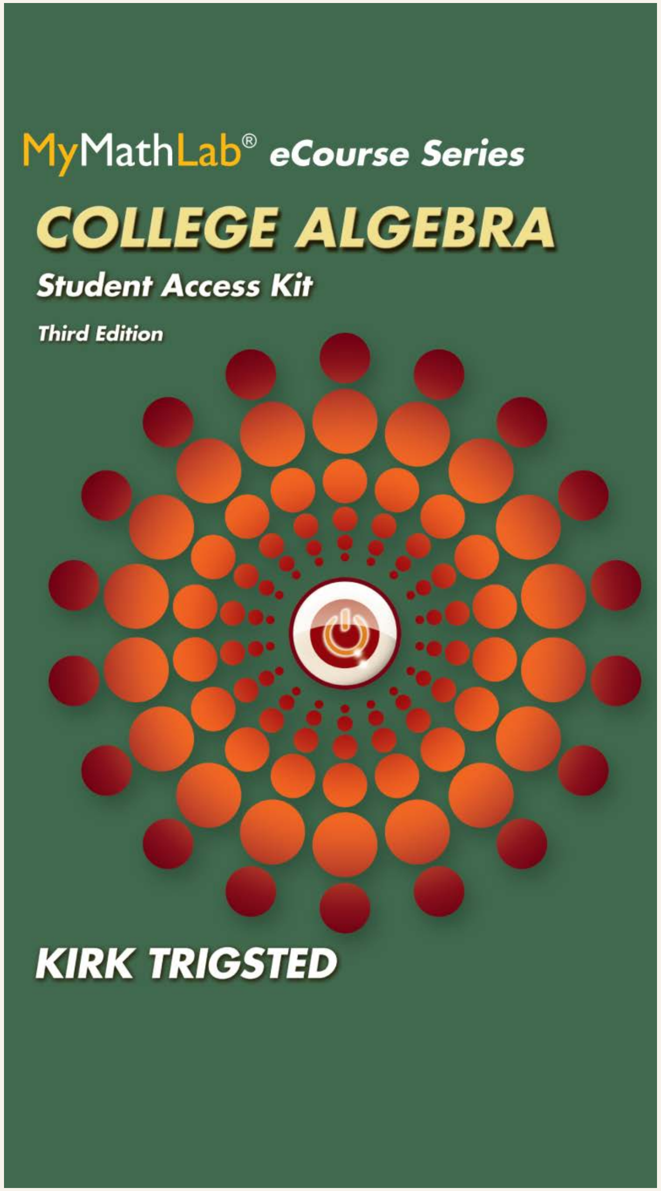
Topic 4.6

Rational Functions and Their Graphs



OBJECTIVES

1. Finding the Domain and Intercepts of Rational Functions
2. Identifying Vertical Asymptotes
3. Identifying Horizontal Asymptotes
4. Using Transformations to Sketch the Graphs of Rational Functions
5. Sketching Rational Functions Having Removable Discontinuities
6. Identifying Slant Asymptotes
7. Sketching Rational Functions

Rational Function

A rational function is a function of the form $f(x) = \frac{g(x)}{h(x)}$, where g and h are polynomial functions such that $h(x) \neq 0$.

Domain of *f*: All real numbers except those for which the denominator is zero.

y-intercept: $f(0)$ provided is it defined

x-intercepts: Can be found by solving the equation $g(x) = 0$ provided $g(x)$ and $h(x)$ do not share a common factor.

Finding the Domain and Intercepts of Rational Functions

EXAMPLE Let $f(x) = \dfrac{x-4}{x^2+x-6}$.

a. Determine the domain of *f*.

$$f(x) = \frac{x-4}{x^2+x-6} = \frac{x-4}{(x+3)(x-2)}$$

$$\{x \mid x \neq -3 \text{ or } x \neq 2\}$$

$$(-\infty,-3)\cup(-3,2)\cup(2,\infty)$$

b. Determine the *y*-intercept (if any).

$$f(0) = \frac{0-4}{(0)^2+0-6} = \frac{2}{3}$$

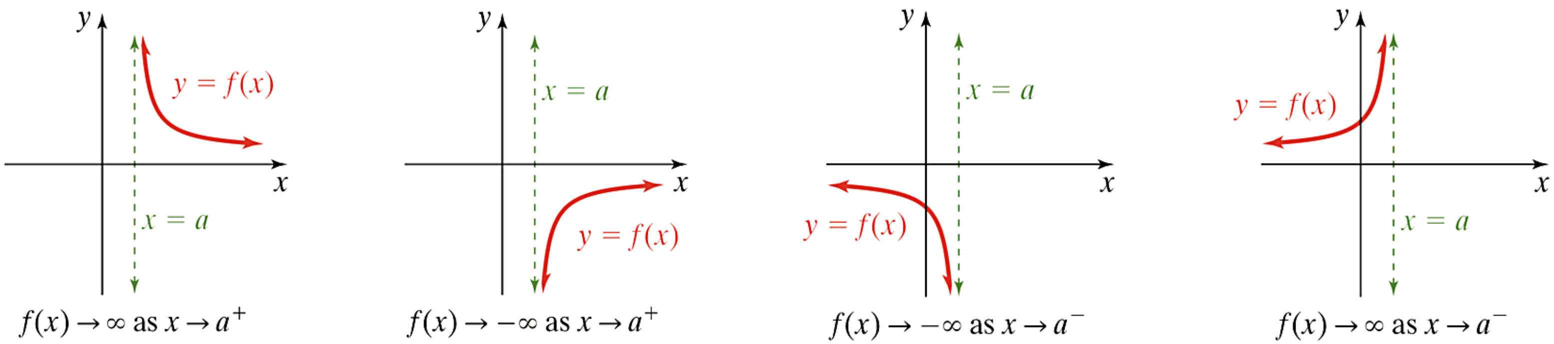
c. Determine *x*-intercepts.

$$x-4=0$$

$$x=4$$

Vertical Asymptote

The vertical line $x = a$ is a vertical asymptote of a function $y = f(x)$ if at least one of the following occurs:



A rational function of the form $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ have no common factors, will have a vertical asymptote at $x = a$ if $h(a) = 0$.

Identifying Vertical Asymptotes

EXAMPLE $f(x) = \dfrac{x-4}{x^2+x-6}$

Find the vertical asymptotes (if any of the function *f*, and then sketch the graph near the vertical asymptotes.

$$f(x) = \frac{x-4}{x^2+x-6} = \frac{x-4}{(x+3)(x-2)}$$ Vertical asymptotes at $x = -3$ and $x = 2$.

Vertical Asymptote	$x = -3$		$x = 2$	
Approach from	Left	Right	Left	Right
Test Value	$x = -3.1$	$x = -2.9$	$x = 1.9$	$x = 2.1$
Substitute Test Value into $f(x) = \frac{x-3}{(x+3)(x-2)}$	$f(-3.1) = \frac{(-3.1-3)}{(-3.1+3)(-3.1-2)} = \frac{(-)}{(-)(-)} = -$	$f(-2.9) = \frac{(-2.9-3)}{(-2.9+3)(-2.9-2)} = \frac{(-)}{(+)(-)} = +$	$f(1.9) = \frac{(1.9-3)}{(1.9+3)(1.9-2)} = \frac{(-)}{(+)(-)} = +$	$f(2.1) = \frac{(2.1-3)}{(2.1+3)(2.1-2)} = \frac{(-)}{(+)(+)} = -$
Result	Sign is negative. $f \to -\infty$	Sign is positive. $f \to \infty$	Sign is positive. $f \to \infty$	Sign is negative. $f \to -\infty$

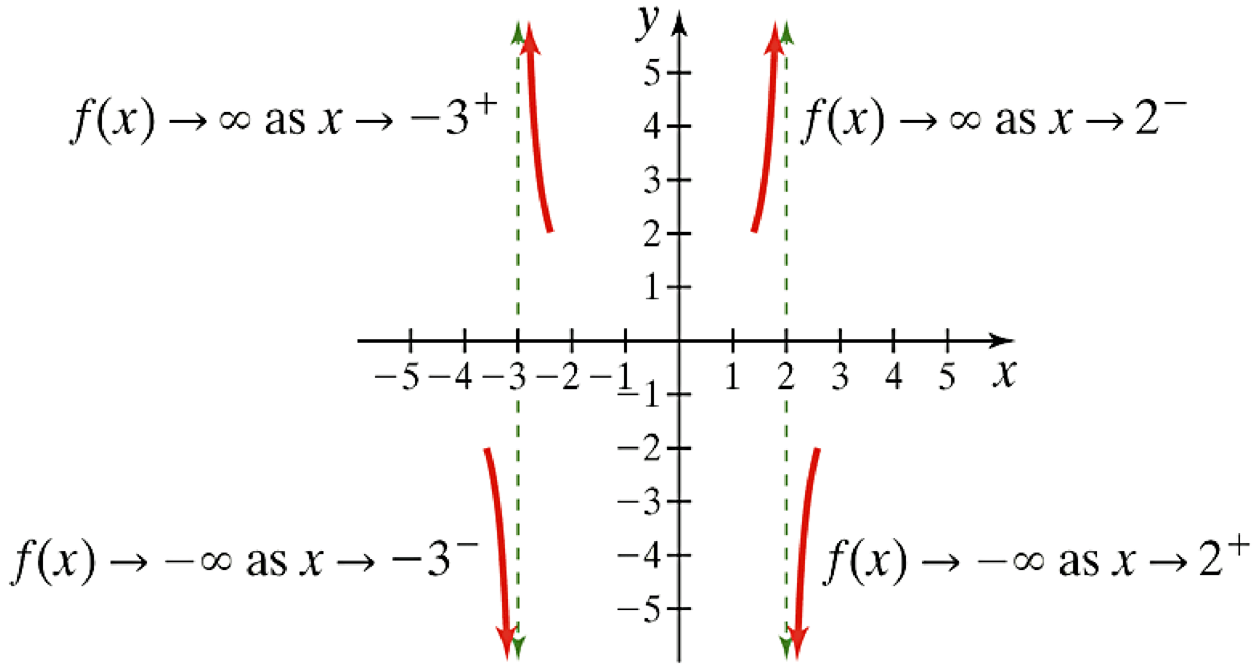
Identifying Vertical Asymptotes

EXAMPLE continued

Find the vertical asymptotes (if any of the function *f*, and then sketch the graph near the vertical asymptotes.

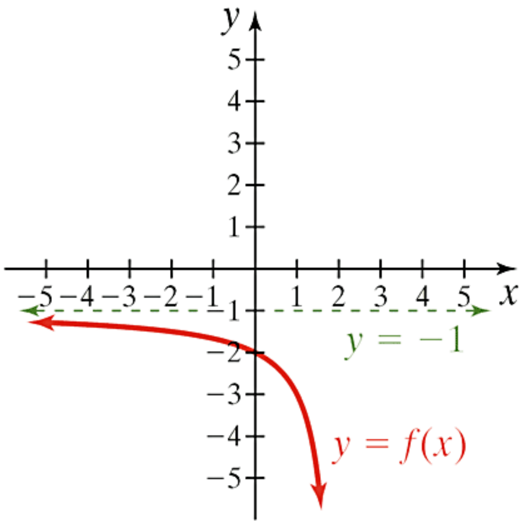
$$f(x) = \frac{x-4}{x^2+x-6} = \frac{x-4}{(x+3)(x-2)}$$

Vertical asymptotes at $x = -3$ and $x = 2$.

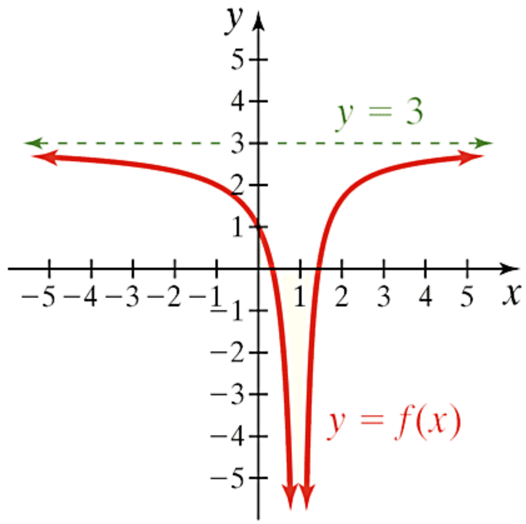


Horizontal Asymptotes

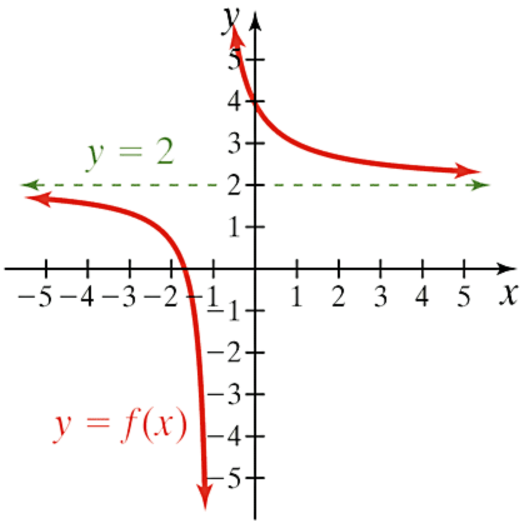
A horizontal line $y = H$ is a horizontal asymptote of a function f if the values of $f(x)$ approach some fixed number H as the values of x approach ∞ or $-\infty$.



The line $y = -1$ is a horizontal asymptote because the values of $f(x)$ approach -1 as x approaches $-\infty$.



The line $y = 3$ is a horizontal asymptote because the values of $f(x)$ approach 3 as x approaches $\pm\infty$.



The line $y = 2$ is a horizontal asymptote because the values of $f(x)$ approach 2 as x approaches $\pm\infty$.

Properties of Horizontal Asymptotes or Rational Functions

- Although a rational function can have many vertical asymptotes, it can have, at most, one horizontal asymptote.
- The graph of a rational function will never intersect a vertical asymptote but may intersect a horizontal asymptote.
- A rational function $f(x) = \dfrac{g(x)}{h(x)}$ that is written in lowest terms (all common factors of the numerator and denominator have been canceled) will have a horizontal asymptote whenever the degree of $h(x)$ is greater than or equal to the degree of $g(x)$.

Finding Horizontal Asymptotes of a Rational Function

Let $f(x) = \dfrac{g(x)}{h(x)} = \dfrac{a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \cdots + b_1 x + b_0}$,

$a_n \neq 0, b_m \neq 0$, where f is written in lowest terms, n is the degree of g, and m is the degree of h.

- If $m > n$, then $y = 0$ is the horizontal asymptote.
- If $m = n$, then the horizontal asymptote is $y = \dfrac{a_n}{b_m}$,

  the ratio of the leading coefficients.
- If $m < n$, tahen there are no horizontal asymptotes.

Identifying Horizontal Asymptotes

EXAMPLE

Find the horizontal asymptote of the graph of each rational function or state that one does not exist.

a. $f(x) = \dfrac{x}{x^2 - 4}$

The degree of the denominator is *greater than* the degree of the numerator. Therefore, the graph has a horizontal asymptote whose equation is $y = 0$.

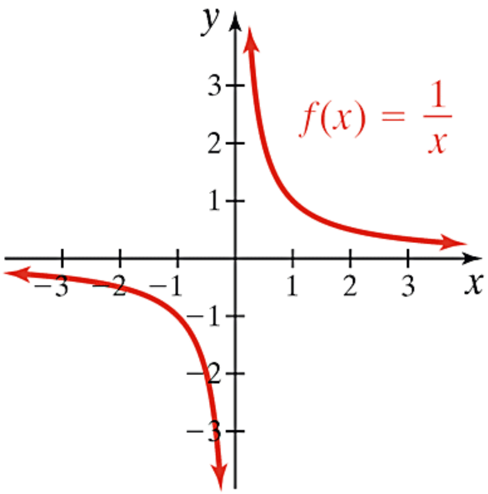
b. $f(x) = \dfrac{4x^2 - x + 1}{1 - 2x^2}$

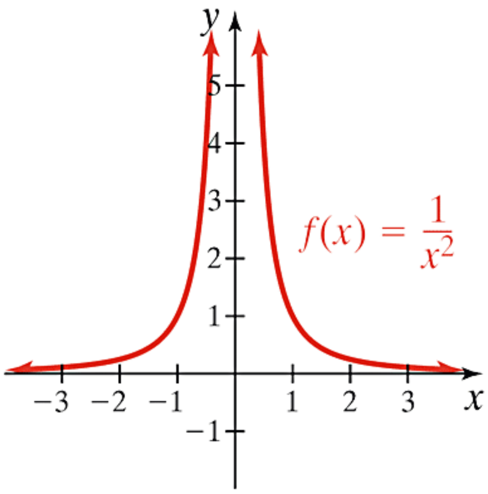
The degree of the denominator is equal to the degree of the numerator. The equation of the horizontal asymptote is $y = 4/-2$ or $y = -2$.

c. $f(x) = \dfrac{2x^3 + 3x^2 - 2x - 2}{x - 1}$

The degree of the denominator is *less than* the degree of the numerator. Therefore, the graph has no horizontal asymptotes.

Graphs of $f(x)=\frac{1}{x}$ and $f(x)=\frac{1}{x^2}$





Properties of the graph of $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

No intercepts

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

Odd function $f(-x) = -f(x)$

The graph is symmetric about the origin.

Properties of the graph of $f(x) = \frac{1}{x^2}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(0, \infty)$

No intercepts

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

Even function $f(x) = f(-x)$

The graph is symmetric about the y-axis.

Using Transformations to Sketch the Graphs of Rational Functions

EXAMPLE

Use transformations to sketch the graph of $f(x) = \dfrac{-2}{(x+3)^2} + 1.$

1. Horizontally shift the graph of $y = \dfrac{1}{x^2}$ to the left three units to obtain the graph of $y = \dfrac{1}{(x+3)^2}.$

2. Vertically stretch the graph of $y = \dfrac{1}{(x+3)^2}$ by a factor of 2 to obtain the graph of $y = \dfrac{2}{(x+3)^2}.$

Using Transformations to Sketch the Graphs of Rational Functions

EXAMPLE continued

Use transformations to sketch the graph of $f(x) = \dfrac{-2}{(x+3)^2} + 1.$

3. Reflect the graph of $y = \dfrac{2}{(x+3)^2}$ about the x-axis

   to obtain the graph of $y = \dfrac{-2}{(x+3)^2}.$

4. Verically shift the graph of $y = \dfrac{-2}{(x+3)^2}$ up one unit

   to obtain the graph of $y = \dfrac{-2}{(x+3)^2} + 1.$

Using Transformations to Sketch the Graphs of Rational Functions

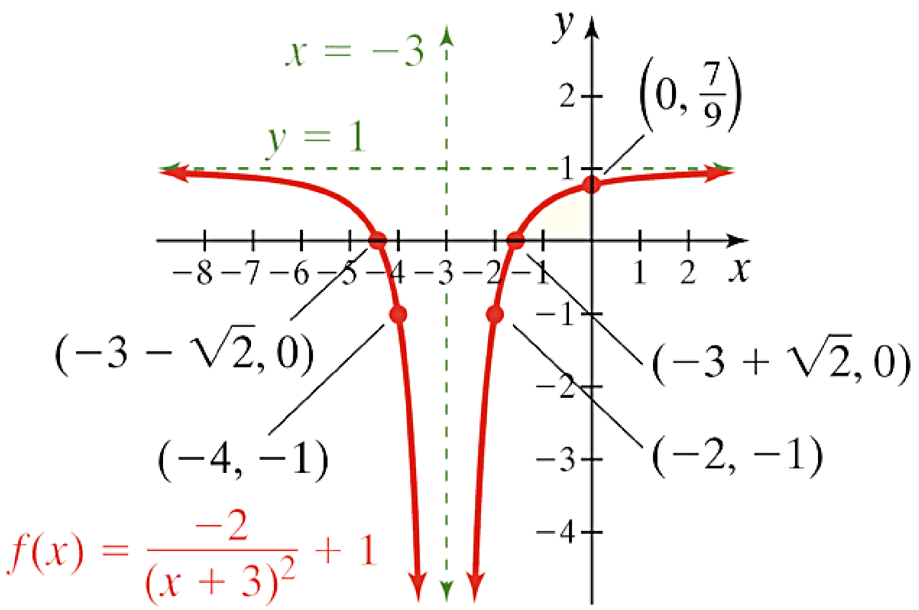
EXAMPLE continued

Use transformations to sketch the graph of $f(x) = \dfrac{-2}{(x+3)^2} + 1.$

The horizontal asymptote is now shifted up one unit to $y = 1$.

The x-intercepts are at $-3 - \sqrt{2}$ and $-3 + \sqrt{2}$.

The y-intercept is $\dfrac{7}{9}$.



Removable Discontinuities

Occur when the numerator and denominator of a rational function have a common factor. The removable discontinuity or "hole" occurs when the common factor equals zero.

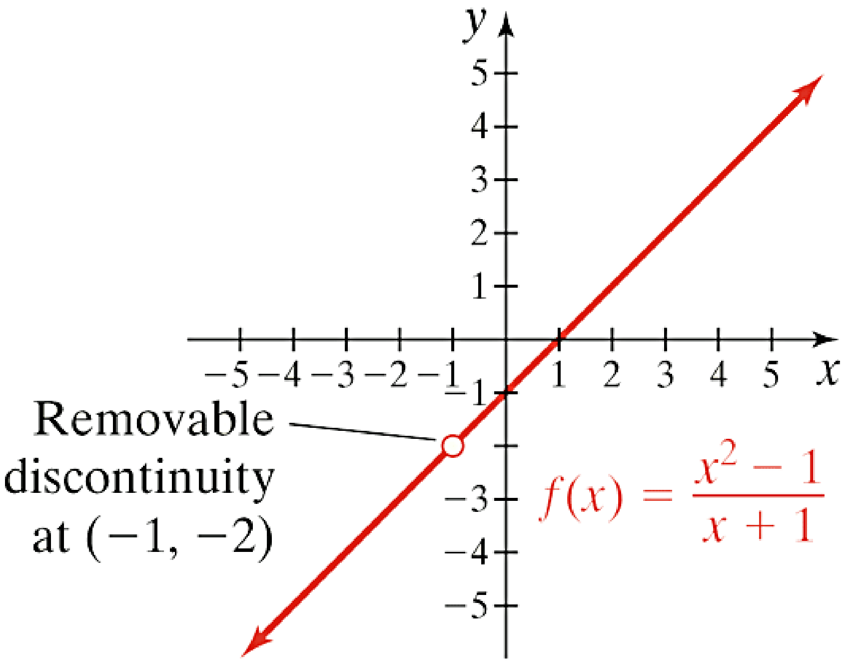
Sketching Rational Functions Having Removable Discontinuities

EXAMPLE

Sketch the graph of $f(x) = \dfrac{x^2 - 1}{x + 1}$, and find the coordinates of all removable discontinuities.

$$f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1}$$
$$= x - 1 \text{ for } x \neq -1$$

The graph is a line with a hole in it at the point (–1, –2).



Sketching Rational Functions Having Removable Discontinuities

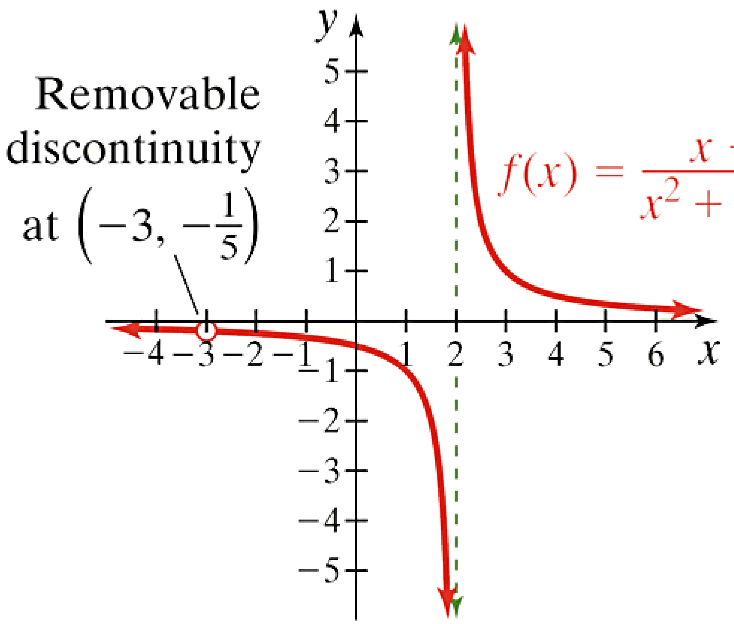
EXAMPLE

Sketch the graph of $f(x) = \dfrac{x+3}{x^2+x-6}$, and find the coordinates of all removable discontinuities.

$$f(x) = \frac{x+3}{x^2+x-6} = \frac{x+3}{(x+3)(x-2)}$$

$$= \frac{1}{x-2} \text{ for } x \neq -3$$

The graph has a hole in it at the point $\left(-3, -\dfrac{1}{5}\right)$.



Slant Asymptotes

- Occur when the degree of the denominator is exactly 1 less than the degree of the numerator.
- To find the slant asymptote: Use synthetic or long division to rewrite f. The slant asymptote is $y =$ the quotient of the division.

Identifying Slant Asymptotes

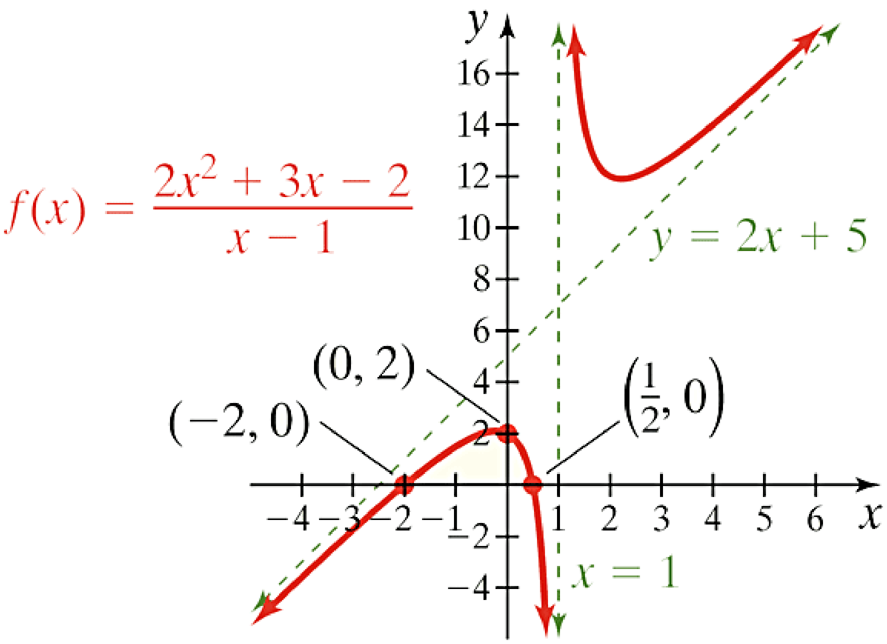
EXAMPLE

Find the slant asymptote of $f(x) = \dfrac{2x^2 + 3x - 2}{x - 1}$.

Using synthetic or long division, we can rewrite f as

$$f(x) = 2x + 5 + \frac{3}{x - 1}$$

so the slant asymptote is the line $y = 2x + 5$.



Sketching Rational Functions

Steps for Graphing Rational Functions of the Form $f(x) = \dfrac{g(x)}{h(x)}$

1. Find the domain.
2. If $g(x)$ and $h(x)$ have common factors, cancel all common factors determining the x-coordinates of any removable discontinuities and rewrite f in lowest terms.
3. Check for symmetry.
   1. If $f(-x) = -f(x)$, then the graph of $f(x)$ is odd and thus symmetric about the origin.
   2. If $f(x) = f(-x)$, then the graph of $f(x)$ is even and thus symmetric about the y-axis.
4. Find the y-intercept by evaluating $f(0)$.

Sketching Rational Functions

Steps for Graphing Rational Functions of the Form $f(x) = \dfrac{g(x)}{h(x)}$

5. Find the x-intercepts by finding the zeros of the numerator of f, being careful to use the new numerator if a common factor has been removed.
6. Find the vertical asymptotes by finding the zeros of the denominator of f, being careful to use the new denominator if a common factor has been removed. Use test values to determine the behavior of the graph on each side of the vertical asymptotes.
7. Determine whether the graph has any horizontal or slant asymptotes
8. Plot point, choosing values of x between each intercept and values of x on either side of all vertical asymptotes.
9. Complete the sketch.

Sketching Rational Functions

EXAMPLE Sketch the graph of $f(x)=\dfrac{x^3+2x^2-9x-18}{x^3+6x^2+5x-12}$.

1. $f(x)=\dfrac{x^3+2x^2-9x-18}{x^3+6x^2+5x-12}=\dfrac{(x+3)(x-3)(x+2)}{(x+3)(x-1)(x+4)}$

   The domain is $\{x \mid x \neq -4, x \neq -3, x \neq 1\}$.

2. $f(x)=\dfrac{(x-3)(x+2)}{(x-1)(x+4)}$ for $x \neq -3$

   The removable discontinuity is at the point $\left(-3, -\dfrac{3}{2}\right)$

3. The function is neither odd nor even because $f(-x) \neq -f(x)$ and $f(-x) \neq f(x)$. Therefore, the graph is not symmetric about the origin or the y-axis.

Sketching Rational Functions

EXAMPLE continued Sketch the graph of $f(x)=\dfrac{x^3+2x^2-9x-18}{x^3+6x^2+5x-12}$.

4. The y-intercept is $f(0)=\dfrac{3}{2}$.

5. The x-intercepts ocur when $(x-3)(x+2)=0$. The x-intercepts $x=-2$ and $x=3$.

6. The equations of the two vertical asymptotes are $x=-4$ and $x=1$.

   As $x\to -4^-$, $f(x)\to\infty$. The graph approaches positive infinity as x approaches -4 from the left.

   As $x\to -4^+$, $f(x)\to -\infty$. The graph approaches negative infinity as x approaches -4 from the right.

Sketching Rational Functions

EXAMPLE continued Sketch the graph of $f(x) = \dfrac{x^3 + 2x^2 - 9x - 18}{x^3 + 6x^2 + 5x - 12}$.

As $x \to 1^-$, $f(x) \to \infty$. The graph approaches positive infinity as x approaches 1 from the left.

As $x \to 1^+$, $f(x) \to -\infty$. The graph approaches negative infinity as x approaches 1 from the right.

7. Because the degree of the denominator is equal to the degree of the numerator, the graph of f has a horizontal asymptote at $y = 1$.

8. We can evaluate the function at several values of x.

$$f(-5) = 4,\ f\left(\frac{1}{2}\right) = \frac{25}{9},\ \text{and } f(2) = -\frac{2}{3}.$$

Sketching Rational Functions

EXAMPLE continued

Sketch the graph of $f(x) = \dfrac{x^3 + 2x^2 - 9x - 18}{x^3 + 6x^2 + 5x - 12}$.

9. The completed graph is sketched below.