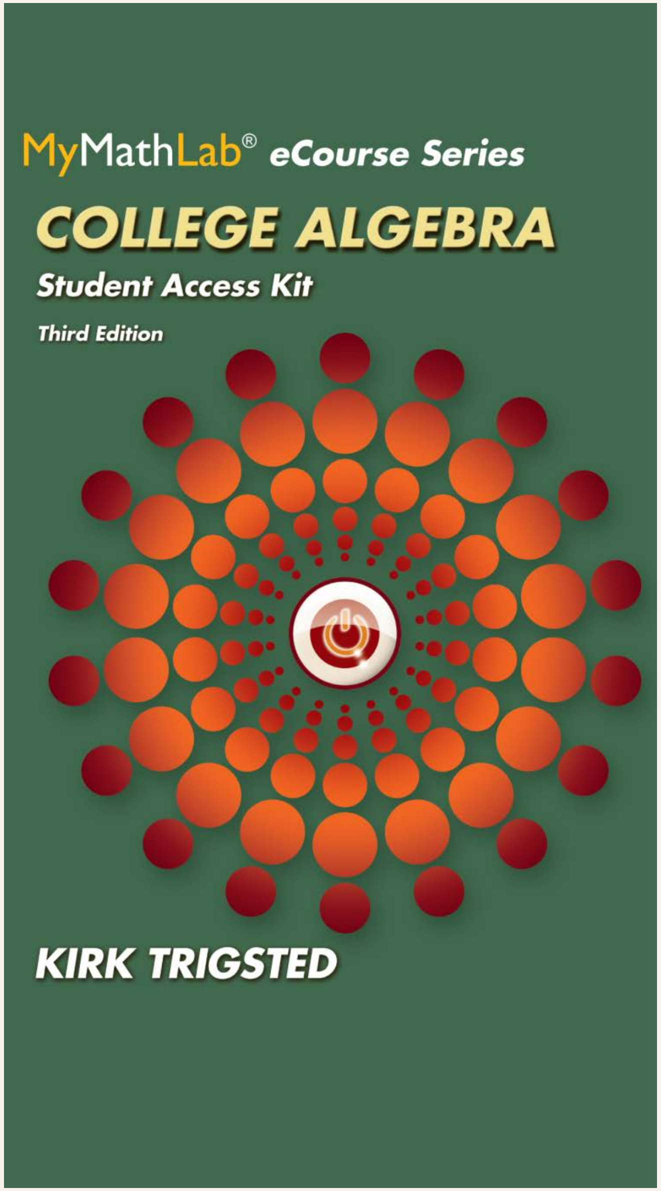
Topic 5.1

Exponential Functions



OBJECTIVES

1. Understanding the Characteristics of Exponential Functions
2. Sketching the Graphs of Exponential Functions Using Transformations
3. Solving Exponential Equations by Relating the Bases
4. Solving Applications of Exponential Functions.

Understanding the Characteristics Of Exponential Functions

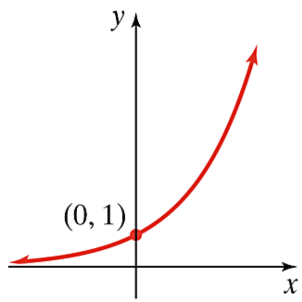
DEFINITION: Exponential Function

An exponential function is a function of the form $f(x) = b^x$ where x is any real number and $b>0$ such that $b \neq 1$. The constant, b, is called the base of the exponential function.

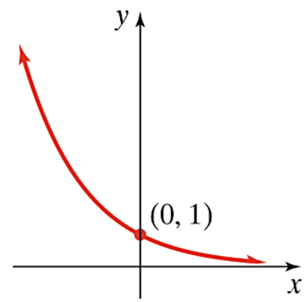
Understanding the Characteristics Of Exponential Functions

Characteristics of Exponential Function

For $b > 0, b \neq 1$, the exponential function with base b is defined by $f(x) = b^x$. The domain of $f(x) = b^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$. The graph of $f(x) = b^x$ has one of the following two shapes:



$f(x) = b^x, b > 1$



$f(x) = b^x, 0 < b < 1$

The graph intersects the y-axis at $(0, 1)$.
$b^x \to \infty$ as $x \to \infty$
$b^x \to 0$ as $x \to -\infty$
The line $y = 0$ is a **horizontal asymptote**.
The function is **one to one**.

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Understanding the Characteristics Of Exponential Functions

EXAMPLE

Evaluate each exponential expression correctly to six decimal places.

a. $\left(\frac{3}{7}\right)^{0.6}$ $\quad$ $\left(\frac{3}{7}\right)^{0.6} \approx 0.601470$

b. $9\left(\frac{5}{8}\right)^{-0.375}$ $\quad$ $9\left(\frac{5}{8}\right)^{-0.375} \approx 10.734640$

c. e^{2} $\quad$ $e^{2} \approx 7.389056$

d. $e^{-0.534}$ $\quad$ $e^{-0.534} \approx 0.586255$

e. $1{,}000e^{0.013}$ $\quad$ $1{,}000e^{0.013} \approx 1{,}013.084867$

Understanding the Characteristics Of Exponential Functions

EXAMPLE

Sketch the graph of the exponential function.

$$f(x) = \left(\tfrac{2}{3}\right)^x \qquad base < 1; \quad \lim_{x \to \infty} y \text{ approaches } 0$$

$$f(-2) = \left(\tfrac{2}{3}\right)^{-2} = \left(\tfrac{3}{2}\right)^{2} = \tfrac{9}{4}$$

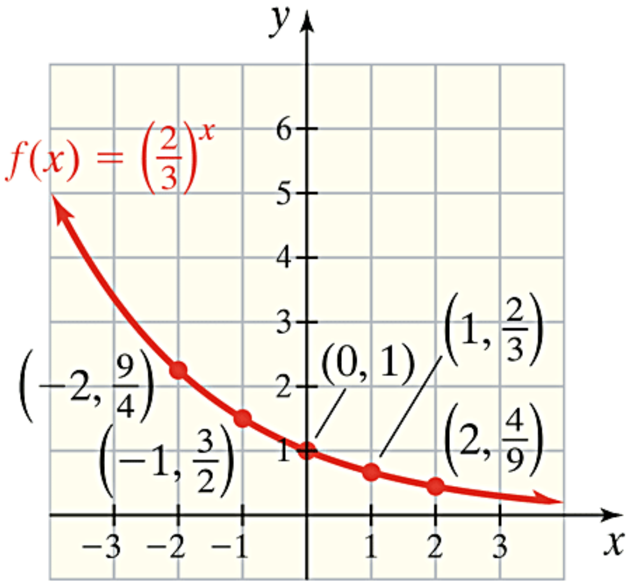
$$f(-1) = \left(\tfrac{2}{3}\right)^{-1} = \left(\tfrac{3}{2}\right)^{1} = \tfrac{3}{2}$$

$$f(1) = \left(\tfrac{2}{3}\right)^{1} = \tfrac{2}{3}$$

$$f(2) = \left(\tfrac{2}{3}\right)^{2} = \tfrac{4}{9}$$

Find points by substituting values

Identify shape based on base



Sketching the Graphs of Exponential Functions Using Transformations

EXAMPLE

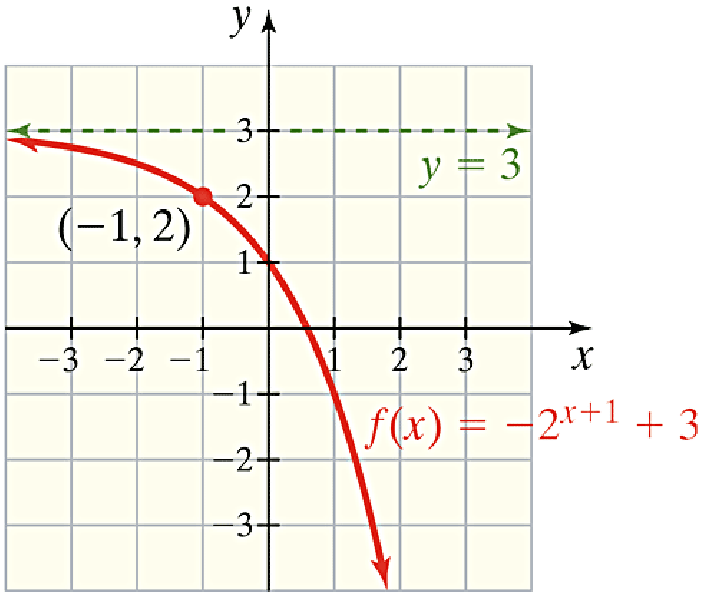
Use transformations to sketch the graph of $f(x) = -2^{x+1} + 3$.

$y = 2^x$ Use basic equation

$y = 2^{x+1}$ Shift left one unit

$y = -2^{x+1}$ Reflect about the *x*-axis

$y = -2^{x+1} + 3$ Shift vertically three units



Solving Exponential Equations by Relating the Bases

Method of Relating the Bases

If an exponential equation can be written in the form $b^u = b^v$, then $u = v$.

Solving Exponential Equations by Relating the Bases

EXAMPLE Solve the following equations:

a. $8 = \frac{1}{16^x}$

$8 = 16^{(-x)}$ Rewrite

$2^3 = \left(2^4\right)^{(-x)}$ Use factors to relate exponents

$2^3 = 2^{-4x}$ Solve

$3 = -4x$

$-\frac{3}{4} = x$

b. $\frac{1}{27^x} = \left(\sqrt[4]{3}\right)^{x-2}$

$27^{-x} = 3^{\frac{x-2}{4}}$ Rewrite

$3^{-3x} = 3^{\frac{x-2}{4}}$ Use factors to relate exponents

$-3x = \frac{x-2}{4}$ Solve

$-12x = x - 2$

$-13x = -2$

$x = \frac{2}{13}$

Solving Applications of Exponential Functions

EXAMPLE

Most golfers find that their golf skills improve dramatically at first and then level off quickly. For example, suppose that the distance (in yards) that a typical beginning golfer can hit a 3-wood after t weeks of practice on the driving range is given by the exponential function $d(t) = 225 - 100e^{-0.7t}$.

This function has been developed after many years of gathering data on beginning golfers. How far can a typical beginning golfer initially hit a 3-wood? How far can a typical beginning golfer hit a 3-wood after 1 week of practice? After 5 weeks? After 9 weeks? Round to the nearest hundredth yard.

$$d(0) = 225 - 100e^{0}$$

Initial driving distance

$$= 225 - 100 \cdot 1$$

$$= 125$$

$$d(1) = 225 - 100e^{-0.7(1)}$$

Distance after 1 week

$$= 175.34$$

Solving Applications of Exponential Functions

EXAMPLE Continued

Most golfers find that their golf skills improve dramatically at first and then level off quickly. For example, suppose that the distance (in yards) that a typical beginning golfer can hit a 3-wood after t weeks of practice on the driving range is given by the exponential function $d(t) = 225 - 100e^{-0.7t}$.

This function has been developed after many years of gathering data on beginning golfers. How far can a typical beginning golfer initially hit a 3-wood? How far can a typical beginning golfer hit a 3-wood after 1 week of practice? After 5 weeks? After 9 weeks? Round to the nearest hundredth yard.

$$d(5) = 225 - 100e^{-0.7(5)}$$ Distance after 5 weeks

$$= 221.98$$

$$d(9) = 225 - 100e^{-0.7(9)}$$ Distance after 9 weeks

$$= 224.82$$

Solving Applications of Exponential Functions

Period Compound Interest Formula

Periodic compound interest can be calculated using the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where
P= Principal
A= Total amount after t years
r= Interest rate per year
n= Number of times interest is compounded yearly
t= Number of years

Solving Applications of Exponential Functions

EXAMPLE

Which investment results in the greatest total amount after 25 years?
Investment A: $12,000 invested at 3% compounded monthly
Investment B: $10,000 invested at 3.9% compounded quarterly

Investment A: $P = 12,000; r = 0.03; n = 12; t = 25$

$$A = 12,000\left(1 + \frac{0.03}{12}\right)^{12(25)} \approx \$25,380.23$$

Investment B: $P = 10,000; r = 0.039; n = 4; t = 25$

$$A = 10,000\left(1 + \frac{0.039}{4}\right)^{4(25)} \approx \$26,386.77$$

Investment B will yield the most money after 25 years.

Solving Applications of Exponential Functions

Continuous Compound Interest

Continuous compound interest can be calculated using the formula

Where $A = Pe^{rt}$

A= Total amount after t years

P= Principal

r= Interest rate per year

t= Number of years

Solving Applications of Exponential Functions

EXAMPLE

How much money would be in an account after 5 years if an original investment of $6,000 was compounded continuously at 4.5%? Compare this amount to the same investment that was compounded daily. Round to the nearest cent

$$A = 6{,}000e^{0.45(5)} \approx \$7{,}513.94$$ compounded continuously

$$A = 6{,}000\left(1 + \frac{0.045}{365}\right)^{365(5)} \approx \$7{,}513.83$$ compounded daily

Continuous compound interest yields $.11 more interest after 5 years.

Solving Applications of Exponential Functions

Present Value Formula for Period Compound Interest

Present value for periodic compound interest can be calculated using the formula

$$P = A\left(1+\frac{r}{n}\right)^{-nt}$$

Where

P= Principal

A= Total amount after t years

r= Interest rate per year

n= Number of times interest is compounded yearly

t= Number of years

Solving Applications of Exponential Functions

Present Value Formula for Continuous Compound Interest

The present value of *A* dollars after *t* years of continuous compound interest, with interest rate *r*, is given by the formula

$$P = Ae^{-rt}$$

Solving Applications of Exponential Functions

EXAMPLE

a. Find the present value of $8,000 if interest is paid at a rate of 5.6% compounded quarterly for 7 years. Round to the nearest cent.

$$A = 8{,}000; r = 0.056; n = 4; t = 7$$

$$P = A\left(1 + \frac{r}{n}\right)^{-nt} = 8{,}000\left(1 + \frac{0.056}{4}\right)^{-4(7)} \approx \$5{,}420.35$$

Therefore, the present value of $8,000 in 7 years at 5.6% compounded quarterly is $5,420.35.

b. Find the present value of $18,000 if interest is paid at a rate of 8% compounded continuously for 20 years. Round to the nearest cent.

$$A = 18{,}000; r = 0.08; t = 20 \qquad P = Ae^{-rt} = 18{,}000e^{-(.08)(20)} \approx \$3{,}634.14$$

Solving Applications of Exponential Functions

Exponential Growth

A model that describes the population, *P*, after a certain time, *t*, is

$$P(t) = P_0 e^{kt}$$

where $P_0 = P(0)$ is the initial population and $k > 0$ is a constant called the **relative growth rate**. (*Note*: *k* may be given as a percent.)

Solving Applications of Exponential Functions

EXAMPLE

The population of a small town follows the exponential growth model $P(t) = 900e^{.015t}$, where t is the number of years after 1900.
Answer the following questions, rounding each answer to the nearest whole number:

a. What was the population of this town in 1900?

$$P(0) = 900e^{0.015(0)} = 900$$

b. What was the population of this town in 1950?

$$P(50) = 900e^{0.015(50)} \approx 1{,}905$$

c. Use this model to predict the population of this town in 2012.

$$P(112) = 900e^{0.015(112)} \approx 4{,}829$$