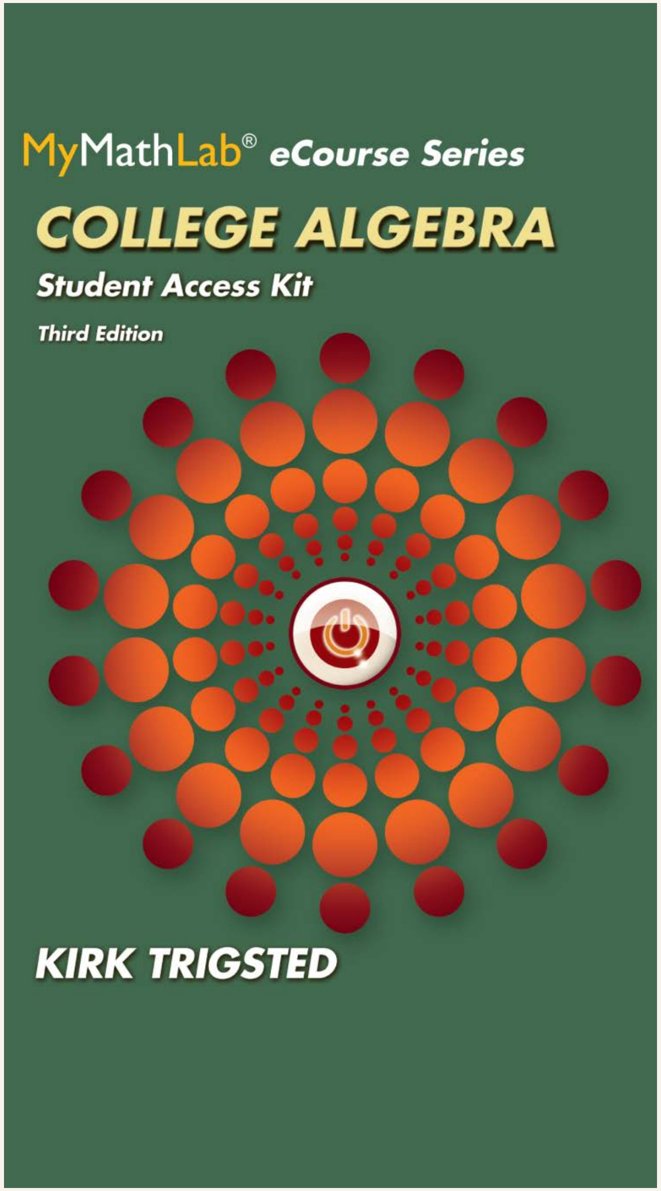
Topic 5.5

Applications of Exponential and Logarithmic Functions



OBJECTIVES

1. Solving Compound Interest Applications
2. Exponential Growth and Decay
3. Solving Logistic Growth Applications
4. Using Newton’s Law of Cooling

Solving Compound Interest Applications

Compound Interest Formulas

- Periodic Compound Interest Formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

- Continuous Compound Interest Formula

$$A = Pe^{rt}$$

where
A= Total amount after t years
P= Principal (original investment)
r= Interest rate per year
n= Number of times interest is compounded per year
t= Number of years

Solving Compound Interest Applications

EXAMPLE

How long will it take (in years and months) for an investment to double if it earns 7.5% compounded monthly?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$2P = P\left(1 + \frac{.075}{12}\right)^{12t}$$

$$2 = \left(1 + \frac{.075}{12}\right)^{12t}$$

$$2 = (1.00625)^{12t}$$

$$\ln 2 = \ln (1.00625)^{12t}$$

$$\ln 2 = 12t \ln(1.00625)$$

$$t = \frac{\ln 2}{12 \ln(1.00625)}$$

$$t \approx 9.27 \text{ years}$$

$$.27 \text{ years} \cdot \frac{12 \text{ months}}{1 \text{ year}} = 3.24 \text{ months}$$

9 years and 4 months

Solving Compound Interest Applications

EXAMPLE

Suppose an investment of \$5,000 compounded continuously grew to an amount of \$5,130.50 in 6 months. Find the interest rate, and then determine how long it will take for the investment to grow to \$6,000. Round the interest rate to the nearest hundredth of a percent and the time to the nearest hundredth of a year.

$$A = Pe^{rt}$$

$$5{,}130.50 = 5{,}000e^{r(0.5)}$$

$$\ln\left(\frac{5{,}130.50}{5{,}000}\right) = \ln e^{0.5r}$$

$$\ln\left(\frac{5{,}130.50}{5{,}000}\right) = 0.5r$$

$$\frac{\ln\left(\frac{5{,}130.50}{5{,}000}\right)}{0.5} = r \approx .051530$$

interest rate is 5.15%

$$A = Pe^{rt}$$

$$6{,}000 = 5{,}000e^{0.0515t}$$

$$\frac{6}{5} = e^{0.0515t}$$

$$\ln\frac{6}{5} = \ln e^{0.0515t}$$

$$\ln\frac{6}{5} = 0.0515t$$

$$\frac{\ln\frac{6}{5}}{.0515} = t \approx 3.54 \text{ years}$$

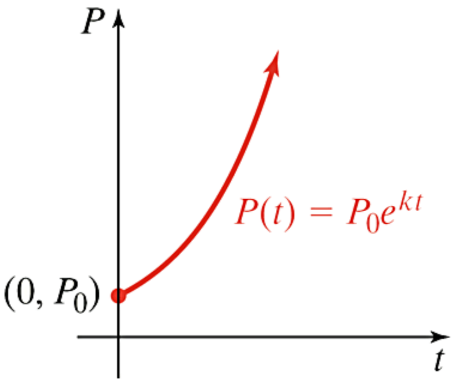
Exponential Growth and Decay

Exponential Growth

A model that describes the exponential uninhibited growth of a population, P, after a certain time, t, is

$$P(t) = P_0 e^{kt}$$

Where $P(t)=P_0e^{kt}$ is the initial population and $k>0$ is a constant called the relative growth rate.

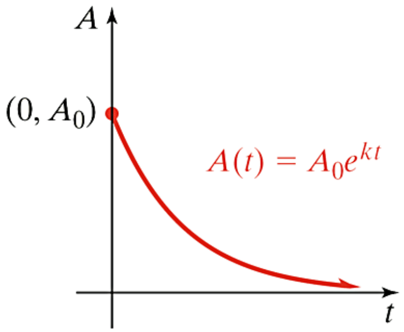


Exponential Decay

A model that describes the exponential decay of a population, quantity, or amount A, after a certain time, t, is

$$A(t) = A_0 e^{kt}$$

Where $A_0=A(0)$ is the initial population and $k<0$ is a constant called the relative decay constant.



Exponential Growth and Decay

EXAMPLE

The population of a small town grows at a rate proportional to its current size. In 1900, the population of the town was 900. In 1920 the population had grown to 1,600. What was the population of this town in 1950?

$$P(t) = P_0 e^{kt}$$

$$P(20) = 900e^{k(20)} = 1{,}600$$

$$e^{20k} = \frac{16}{9}$$

$$\ln e^{20k} = \ln\frac{16}{9}$$

$$20k = \ln\frac{16}{9}$$

$$k = \frac{\ln\frac{16}{9}}{20}$$

$$P(50) = 900e^{\frac{\ln\frac{16}{9}}{20}(50)}$$

$$\approx 3{,}793$$

Exponential Growth and Decay

EXAMPLE

Suppose that a meteorite is found containing 4% of its original krypton-99. If the half-life of krypton-99 is 80 years, how old is the meteorite? Round to the nearest year.

$$A(t) = A_0 e^{kt}$$

$$.5A_0 = A_0 e^{k(80)}$$

$$.5 = e^{k(80)}$$

$$\ln .5 = \ln e^{k(80)}$$

$$\ln .5 = 80k$$

$$\frac{-\ln 2}{80} = k$$

$$.04A_0 = A_0 e^{\frac{-\ln 2}{80}t}$$

$$.04 = e^{\frac{-\ln 2}{80}t}$$

$$\ln .04 = \ln e^{\frac{-\ln 2}{80}t}$$

$$\ln .04 = \frac{-\ln 2}{80}t$$

$$\frac{\ln .04}{\left(\frac{-\ln 2}{80}\right)} = t \approx 372 \text{ years}$$

Solving Logistic Growth Applications

Logistic Growth

A model that describes the logistic growth of a population, P, at any time t, is given by the function

$$P(t)=\frac{C}{1+Be^{kt}}$$

Where B,C, and k are constants with $C>0$ and $k<0$.

Solving Logistic Growth

EXAMPLE

Ten goldfish were introduced into a small pond. Because of limited food, space and oxygen, the carrying capacity of the pond is 400 goldfish. The goldfish population at any time t, in days, is modeled by the logistic growth function $F(t)=\dfrac{C}{1+Be^{kt}}$.

If 30 goldfish are in the pond after 20 days,

a. Find B

$$10=\frac{400}{1+Be^{k(0)}}$$

$$10=\frac{400}{1+B}$$

$$10+10B=400$$

$$B=39$$

Solving Logistic Growth

EXAMPLE continued

Ten goldfish were introduced into a small pond. Because of limited food, space and oxygen, the carrying capacity of the pond is 400 goldfish. The goldfish population at any time t, in days, is modeled by the logistic growth function $F(t) = \frac{C}{1 + Be^{kt}}$.

If 30 goldfish are in the pond after 20 days,

b. Find k

$$30 = \frac{400}{1 + 39e^{k(20)}}$$

$$30(1 + 39e^{20k}) = 400$$

$$30 + 1{,}170e^{20k} = 400$$

$$1{,}170e^{20k} = 370$$

$$e^{20k} = \frac{37}{117}$$

$$\ln e^{20k} = \ln \frac{37}{117}$$

$$20k = \ln \frac{37}{117}$$

$$k = \frac{\ln \frac{37}{117}}{20}$$

Solving Logistic Growth

EXAMPLE continued

Ten goldfish were introduced into a small pond. Because of limited food, space and oxygen, the carrying capacity of the pond is 400 goldfish. The goldfish population at any time t, in days, is modeled by the logistic growth function $F(t) = \dfrac{C}{1 + Be^{kt}}.$
If 30 goldfish are in the pond after 20 days,

c. When will the pond contain 250 goldfish? Round to the nearest whole number.

$$250 = \frac{400}{1 + 39e^{\frac{\ln\frac{37}{117}}{20}t}}$$

$$1 + 39e^{\frac{\ln\frac{37}{117}}{20}t} = \frac{8}{5}$$

$$39e^{\frac{\ln\frac{37}{117}}{20}t} = \frac{3}{5}$$

$$e^{\frac{\ln\frac{37}{117}}{20}t} = \frac{1}{65}$$

$$e^{\ln\left(\frac{37}{117}\right)^{\frac{t}{20}}} = \frac{1}{65}$$

$$\left(\frac{37}{117}\right)^{\frac{t}{20}} = \frac{1}{65}$$

$$\ln\left(\frac{37}{117}\right)^{\frac{t}{20}} = \ln\left(\frac{1}{65}\right)$$

$$\frac{t}{20}\ln\left(\frac{37}{117}\right) = \ln\left(\frac{1}{65}\right)$$

Solving Logistic Growth

EXAMPLE continued

Ten goldfish were introduced into a small pond. Because of limited food, space and oxygen, the carrying capacity of the pond is 400 goldfish. The goldfish population at any time t, in days, is modeled by the logistic growth function $F(t)=\dfrac{C}{1+Be^{kt}}$.
If 30 goldfish are in the pond after 20 days,

c. When will the pond contain 250 goldfish? Round to the nearest whole number.

$$\frac{t}{20}\ln\left(\frac{37}{117}\right)=\ln\left(\frac{1}{65}\right)$$

$$\frac{t}{20}=\frac{\ln\left(\frac{1}{65}\right)}{\ln\left(\frac{37}{117}\right)}$$

$$t=20\frac{\ln\left(\frac{1}{65}\right)}{\ln\left(\frac{37}{117}\right)}$$

≈ 73 days until the pond contains 250 goldfish

Using Newton's Law of Cooling

Newton's Law of Cooling

The temperature T, of an object at any time t, is given by

$$T(s) = S + (T_0 - S)e^{kt}$$

Where T_0 is the original temperature of the object, S is the constant temperature of the surroundings, and k is the cooling constant.

Using Newton's Law of Cooling

EXAMPLE

Suppose that the temperature of a cup of hot tea obeys Newton's law of cooling. If the tea has a temperature of 200ºF in a room that maintains a constant temperature of 69ºF, determine when the tea reaches a temperature of 146ºF. Round to the nearest minute.

$$T(s) = S + (T_0 - S)e^{kt}$$

$$T(s) = 69 + (200 - 69)e^{kt}$$

$$T(s) = 69 + (131)e^{kt}$$

$$189 = 69 + (131)e^{k(1)}$$

$$120 = 131e^{k}$$

$$\frac{120}{131} = e^{k}$$

$$\ln\frac{120}{131} = \ln e^{k}$$

$$\ln\frac{120}{131} = k$$

$$146 = 69 + (131)e^{\ln\frac{120}{131}t}$$

$$77 = 131e^{\ln\frac{120}{131}t}$$

$$\frac{77}{131} = e^{\ln\frac{120}{131}t}$$

$$\ln\frac{77}{131} = \ln e^{\ln\frac{120}{131}t}$$

$$\ln\frac{77}{131} = \ln\frac{120}{131}t$$

$$\frac{\ln\frac{77}{131}}{\ln\frac{120}{131}} = t \approx 6 \text{ minutes}$$