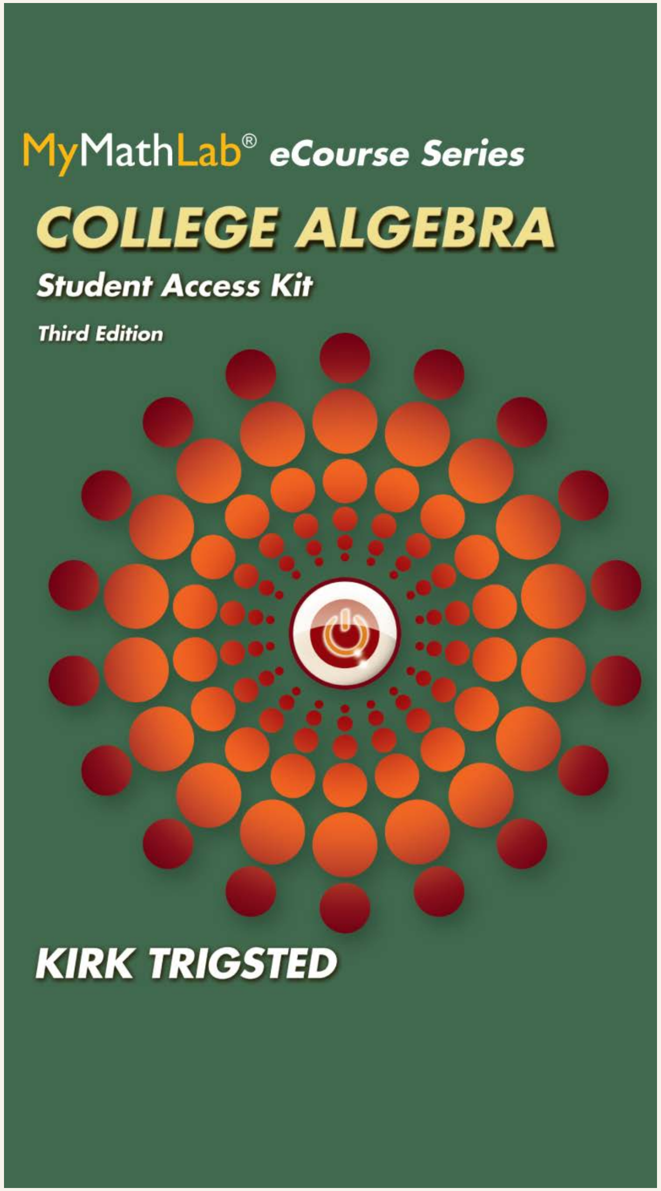
Topic 7.5

Systems of Nonlinear Equations



OBJECTIVES

1. Determining the Number of Solutions to a System of Nonlinear Equations
2. Solving a System of Nonlinear Equations Using the Substitution Method
3. Solving a System of Nonlinear Equations Using Substitution, Elimination or Graphing
4. Solving Applied Problems Using a System of Nonlinear Equations

Determining the Number of Solutions to a System of Nonlinear Equations

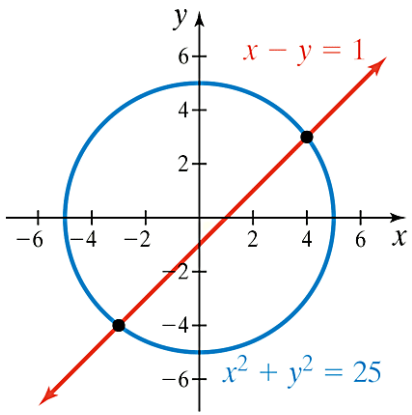
EXAMPLE

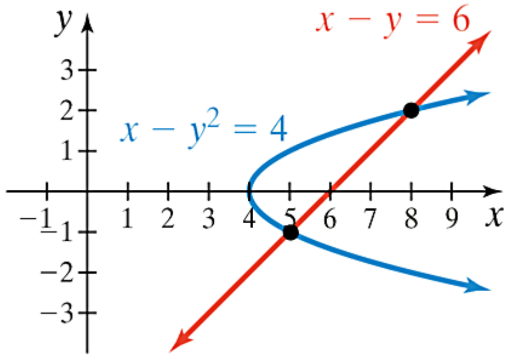
For each system of nonlinear equations, sketch the graph of each equation of the system and then determine the number of real solutions to each system. Do not solve the system.

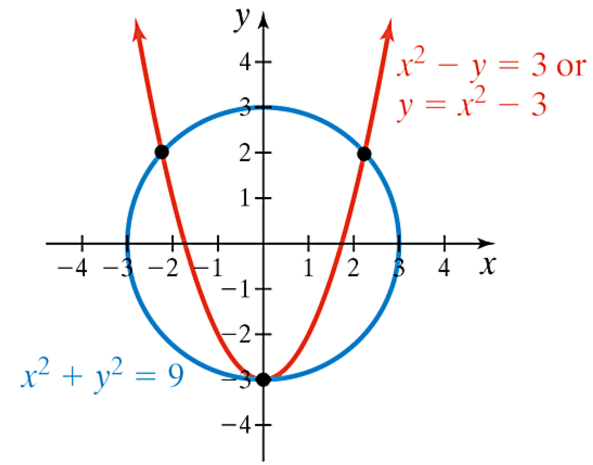
a. $x^2 + y^2 = 25$
$x - y = 1$

b. $x - y^2 = 4$
$x - y = 6$

c. $x^2 + y^2 = 9$
$x^2 - y = 3$







Solving a System of Nonlinear Equations Using the Substitution Method

Solving a System of Nonlinear Equations by the Substitution Method

Step 1. Choose an equation and solve for one variable (or expression) in terms of the other variable.
Step 2. Substitute the expression from step 1 into the other equation.
Step 3. Solve the equation in one variable.
Step 4. Substitute the value(s) found in step 3 into one of the original equations to find the value(s) of the other variable.
Step 5. Check each solution by substituting all proposed solutions into the other equation of the system.

Solving a System of Nonlinear Equations Using the Substitution Method

EXAMPLE

Determine the real solutions to the following system using the substitution method.

$$x^2 + y^2 = 25$$
$$x - y = 1$$

1. $x = y + 1$

2. $\left(y+1\right)^2 + y^2 = 25$

3. $\left(y+1\right)^2 + y^2 = 25$

$$y^2 + 2y + 1 + y^2 = 25$$
$$2y^2 + 2y - 24 = 0$$
$$y^2 + y - 12 = 0$$
$$(y-3)(y+4) = 0$$
$$y = -4 \text{ or } y = 3$$

Solving a System of Nonlinear Equations Using the Substitution Method

EXAMPLE continued

Determine the real solutions to the following system using the substitution method.

$$x^2 + y^2 = 25$$
$$x - y = 1$$

4. $y = -4; x^2 + y^2 = 25$

$$x^2 + (-4)^2 = 25$$
$$x^2 + 16 = 25$$
$$x^2 = 9$$
$$x = \pm 3$$

$(3, -4)$ and $(-3, -4)$

$y = 3; x^2 + y^2 = 25$

$$x^2 + (3)^2 = 25$$
$$x^2 + 9 = 25$$
$$x^2 = 16$$
$$x = \pm 4$$

$(4, 3)$ and $(-4, 3)$

Solving a System of Nonlinear Equations Using the Substitution Method

EXAMPLE continued

Determine the real solutions to the following system using the substitution method.

$$x^2 + y^2 = 25$$
$$x - y = 1$$

5.

$(3, -4)$: $3 - (-4) \stackrel{?}{=} 1 \longrightarrow 3 + 4 \stackrel{?}{=} 1 \longrightarrow 7 \stackrel{?}{=} 1$ ✗ $\longrightarrow (3, -4)$ is **not** a solution.

$(-3, -4)$: $-3 - (-4) \stackrel{?}{=} 1 \longrightarrow -3 + 4 \stackrel{?}{=} 1 \longrightarrow 1 \stackrel{?}{=} 1$ ✓ $\longrightarrow (-3, -4)$ is a solution.

$(4, 3)$: $4 - (3) \stackrel{?}{=} 1 \longrightarrow 4 - 3 \stackrel{?}{=} 1 \longrightarrow 1 \stackrel{?}{=} 1$ ✓ $\longrightarrow (4, 3)$ is a solution.

$(-4, 3)$: $-4 - (3) \stackrel{?}{=} 1 \longrightarrow -4 - 3 \stackrel{?}{=} 1 \longrightarrow -7 \stackrel{?}{=} 1$ ✗ $\longrightarrow (-4, 3)$ is **not** a solution.

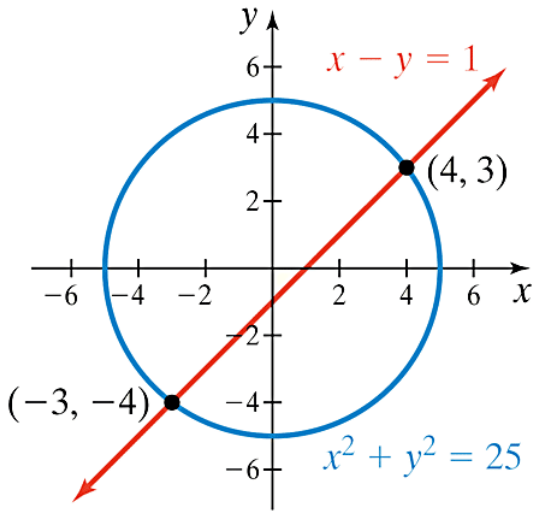
Only two solutions to the nonlinear system are (-3, -4) and (4, 3).

Solving a System of Nonlinear Equations Using the Substitution Method

EXAMPLE continued

Determine the real solutions to the following system using the substitution method.

$$x^2 + y^2 = 25$$
$$x - y = 1$$



The solutions to the nonlinear system $\begin{aligned} x^2 + y^2 &= 25 \\ x - y &= 1 \end{aligned}$ are the ordered pairs $(-3, -4)$ and $(4, 3)$.

It is absolutely critical to always check the solutions to a system of nonlinear equations by substituting each proposed solution into both equations of the system. In the previous example, there were four proposed solutions to the nonlinear system, but only two of those solutions checked in both equations.

Solving a System of Nonlinear Equations Using Substitution, Elimination, or Graphing

Solving a System of Nonlinear Equations by the Elimination Method

Step 1. Choose a variable to eliminate.
Step 2. Multiply one or both equations by an appropriate nonzero constant so that the sum of the coefficients of one of the terms of both equations is zero.
Step 3. Add the two equations together to obtain an equation in one variable.
Step 4. Solve the equation in one variable.
Step 5. Substitute the value(s) obtained from step 4 into one of the original equations to solve for the other variable.
Step 6. Check each solution by substituting all proposed solutions into the other equation of the system.

Solving a System of Nonlinear Equations Using Substitution, Elimination, or Graphing

EXAMPLE

Determine the real solutions to the following system.

$$x^2 + y^2 = 9$$
$$x^2 - y = 3$$

Step 1. Note that both equations contain an x^2 -term. Thus, we can eliminate the variable x.

Step 2. Multiply both sides of the first equation by -1.

$$-1\left(x^2 + y^2 = 9\right) \to -x^2 - y^2 = -9$$
$$x^2 - y = 3 \to \quad x^2 - y \quad = \quad 3$$

Step 3. Now add the two new equations together:

$$\begin{array}{r} -x^2 - y^2 = -9 \\ x^2 - y \quad = \;\; 3 \\ \hline -y^2 - y = -6 \end{array}$$

Solving a System of Nonlinear Equations Using Substitution, Elimination, or Graphing

EXAMPLE continued

Determine the real solutions to the following system.

$$x^2 + y^2 = 9$$
$$x^2 - y = 3$$

Step 4. Solve.

$$-y^2 - y = -6$$
$$y^2 + y - 6 = 0$$
$$(y+3)(y-2) = 0$$
$$y = -3 \text{ or } y = 2$$

Step 5. Solve for x by substituting y = -3 and y = 2 into one of the original equations. Here, we choose the second equation.

$$y = -3 \qquad\qquad y = 2$$
$$x^2 - (-3) = 3 \qquad\qquad x^2 - (2) = 3$$
$$x^2 + 3 = 3 \qquad\qquad x^2 - 2 = 3$$
$$x^2 = 0 \qquad\qquad x^2 = 5$$
$$x = 0 \qquad\qquad x = \pm\sqrt{5}$$

$$(0, -3), (\sqrt{5}, 2), (-\sqrt{5}, 2)$$

Solving a System of Nonlinear Equations Using Substitution, Elimination, or Graphing

EXAMPLE continued

Determine the real solutions to the following system.

$$x^2 + y^2 = 9$$
$$x^2 - y = 3$$

Step 6. Check $(0,-3), (\sqrt{5},2), (-\sqrt{5},2)$.

$$(0)^2 + (-3)^2 = 9 \rightarrow 9 = 9$$
$$(0)^2 - (-3) = 3 \rightarrow 3 = 3$$

$$\left(\sqrt{5}\right)^2 + (2)^2 = 9 \rightarrow 5 + 4 = 9 \rightarrow 9 = 9$$
$$\left(\sqrt{5}\right)^2 - (2) = 3 \rightarrow 5 - 2 = 3 \rightarrow 3 = 3$$
$$\left(-\sqrt{5}\right)^2 + (2)^2 = 9 \rightarrow 5 + 4 = 9 \rightarrow 9 = 9$$
$$\left(-\sqrt{5}\right)^2 - (2) = 3 \rightarrow 5 - 2 = 3 \rightarrow 3 = 3$$

$(0,-3), (\sqrt{5},2)$ and $(-\sqrt{5},2)$ are all solutions.

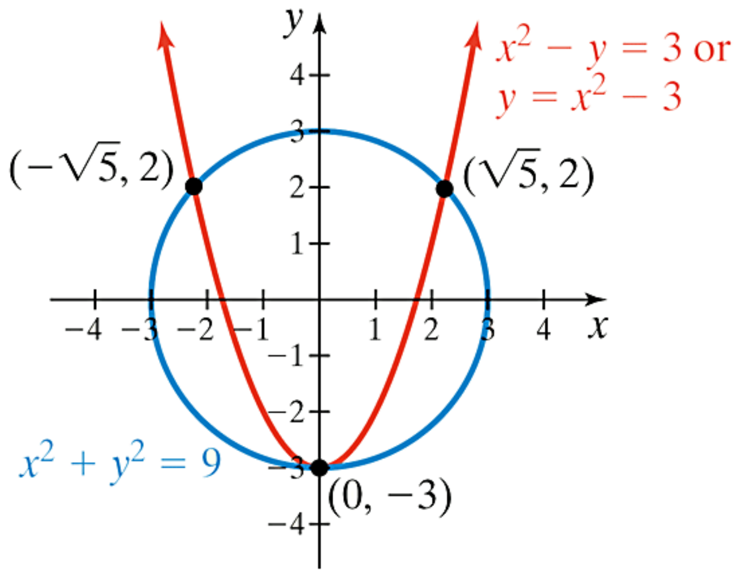
Solving a System of Nonlinear Equations Using Substitution, Elimination, or Graphing

EXAMPLE continued

Determine the real solutions to the following system.

$$x^2 + y^2 = 9$$
$$x^2 - y = 3$$

Step 6. Check $(0,-3), (\sqrt{5}, 2), (-\sqrt{5}, 2)$.



$(0,-3), (\sqrt{5}, 2)$ and $(-\sqrt{5}, 2)$ are all solutions.

Solving a System of Nonlinear Equations Using Substitution, Elimination, or Graphing

EXAMPLE

Determine the real solutions to the following system.

$$y = \log(3x+1) - 5$$
$$y = \log(x-2) - 4$$

$$\log(3x+1) - 5 = \log(x-2) - 4$$
$$\log(3x+1) - \log(x-2) = 5 - 4$$
$$\log\left(\frac{3x+1}{x-2}\right) = 1$$
$$\frac{3x+1}{x-2} = 10^1$$
$$3x+1 = 10(x-2)$$
$$3x+1 = 10x - 20$$
$$-7x = -21$$
$$x = 3$$

$$y = \log(3-2) - 4$$
$$y = \log(1) - 4$$
$$y = -4$$

check $(3,-4)$

$$-4 = \log(3(3)+1) - 5$$
$$-4 = \log(10) - 5$$
$$-4 = 1 - 5$$
$$-4 = -4$$

check $(3,-4)$

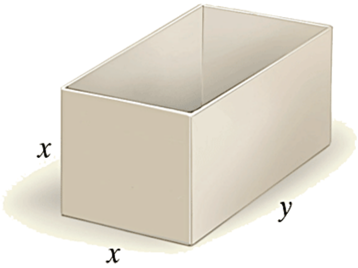
$$-4 = \log(3-2) - 4$$
$$-4 = \log(1) - 4$$
$$-4 = -4$$

$(3,-4)$ is the only solution.

Solving Applied Problems Using a System of Nonlinear Equations

EXAMPLE

An open box (with no lid) has a rectangular base. The height of the box is equal in length to the shortest side of the base. What are the dimensions of the box if the volume is 176 cubic inches and the surface area is 164 square inches?



Draw a box with a rectangular base and label the lengths of the sides of the base x and y, where x represents the shorter of the two sides. Then, the height of the box is x inches.

The surface area of the box is the equal to the sum of the areas of the five sides of the box, which is given to be 164 square inches.

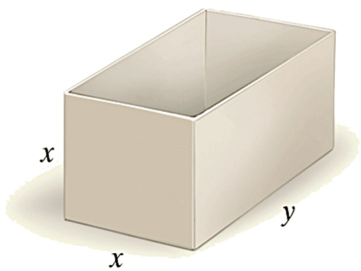
$$x^2 + x^2 + xy + xy + xy = 164$$

$$2x^2 + 3xy = 164$$

Solving Applied Problems Using a System of Nonlinear Equations

EXAMPLE

An open box (with no lid) has a rectangular base. The height of the box is equal in length to the shortest side of the base. What are the dimensions of the box if the volume is 176 cubic inches and the surface area is 164 square inches?



The volume of the box is 176 cubic inches, which is equal to the length times the width times the height.

$$x^2 y = 176$$

$$\begin{aligned} 2x^2 + 3xy &= 164 \\ x^2 y &= 176 \end{aligned}$$

$$y = \frac{176}{x^2}$$

$$2x^2 + 3x\left(\frac{176}{x^2}\right) = 164$$

$$2x^2 + \frac{528}{x} = 164$$

$$2x^3 + 528 = 164x$$

$$2x^3 - 164x + 528 = 0$$

$$x^3 - 182x + 264 = 0$$

Solving Applied Problems Using a System of Nonlinear Equations

EXAMPLE

An open box (with no lid) has a rectangular base. The height of the box is equal in length to the shortest side of the base. What are the dimensions of the box if the volume is 176 cubic inches and the surface area is 164 square inches?

$$x^3 - 182x + 264 = 0$$

Use the **rational zeros theorem** and synthetic division to verify that x - 4 is a factor.

$$\begin{array}{r|rrrr} 4 & 1 & 0 & -82 & 264 \\ & \downarrow & 4 & 16 & -264 \\ \hline & 1 & 4 & -66 & 0 \end{array}$$

The remainder is 0 when $x^3 - 82x + 264$ is divided by $x - 4$.
Therefore, $x - 4$ is a factor of $x^3 - 82x + 264$.

$$(x-4)(x^2+4x-66) = 0$$

$$x - 4 = 0 \text{ or } x^2 + 4x - 66 = 0$$

$$x = 4 \text{ or } x = -2 \pm \sqrt{70}$$

Solving Applied Problems Using a System of Nonlinear Equations

EXAMPLE

An open box (with no lid) has a rectangular base. The height of the box is equal in length to the shortest side of the base. What are the dimensions of the box if the volume is 176 cubic inches and the surface area is 164 square inches?

$$x = 4;$$
$$y = \frac{176}{(4)^2} = \frac{176}{16} = 11$$

$$x = -2 + \sqrt{70} \approx 6.37;$$
$$y = \frac{176}{\left(-2 + \sqrt{70}\right)^2} \approx 4.34$$

As stated in the original problem, the value of x had to be less than the value of y. Therefore, the only possible solution is $x = 4$ and $y = 11$.

check (4,11)

$$2x^2 + 3xy = 164$$
$$x^2 y = 176$$

$$2(4)^2 + 3(4)(11) = 164$$
$$32 + 132 = 164$$
$$164 = 164$$

$$x^2 y = 176$$
$$(4)^2 (11) = 176$$
$$16 \cdot 11 = 176$$
$$176 = 176$$

The dimensions of the box are 4 inches by 11 inches by 4 inches.