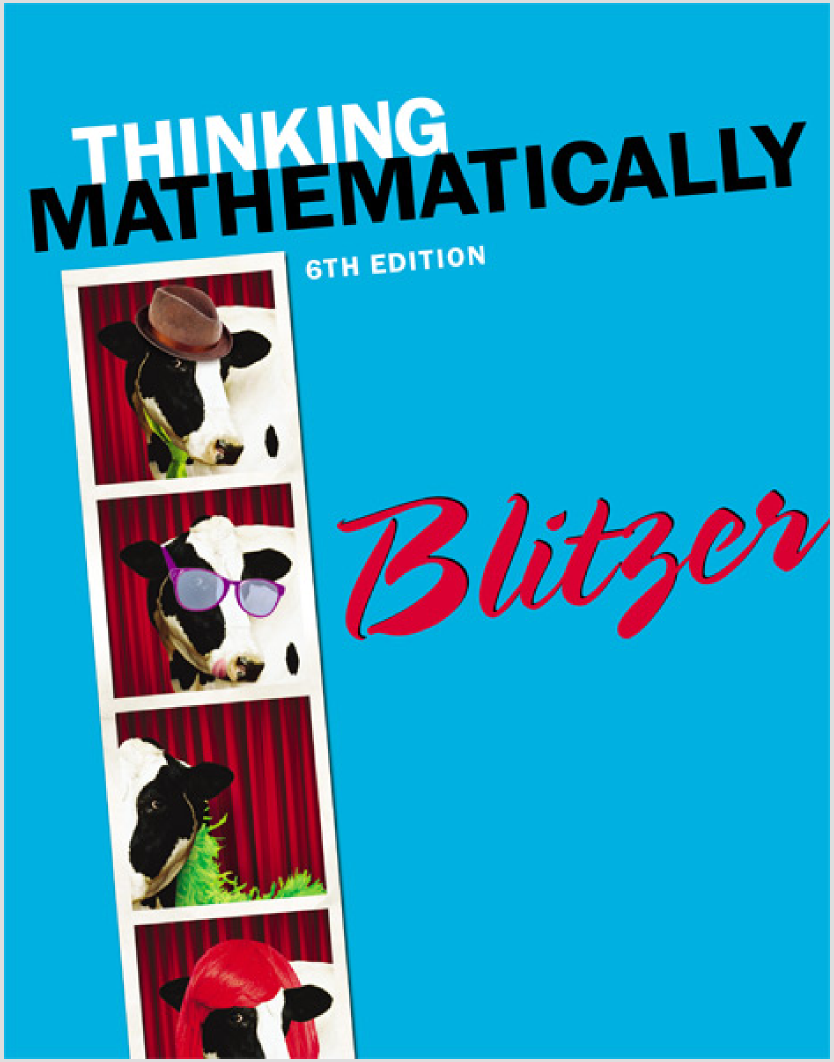
CHAPTER 2

Set Theory



2.1

Basic Set Concepts

Objectives

1. Use three methods to represent sets.
2. Define and recognize the empty set.
3. Use the symbols $\in$ and $\notin$.
4. Apply set notation to sets of natural numbers.
5. Determine a set's cardinal number.
6. Recognize equivalent sets.
7. Distinguish between finite and infinite sets.
8. Recognize equal sets.

Sets

A collection of objects whose contents can be clearly determined.

Elements or **members** are the objects in a set.

A set must be **well-defined**, meaning that its contents can be clearly determined.

The order in which the elements of the set are listed is not important.

Methods for Representing Sets

Capital letters are generally used to name sets.

Word description: Describing the members:

Set *W* is the set of the days of the week.

Roster method: Listing the members:

W = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}.

Commas are used to separate the elements of the set.

Braces, { }, are used to designate that the enclosed elements form a set.

Example: Representing a Set Using a Description

Write a word description of the set:

P = {Washington, Adams, Jefferson, Madison, Monroe}.

Solution

Set P is the set of the first five presidents of the United States.

Example: Representing a Set Using the Roster Method

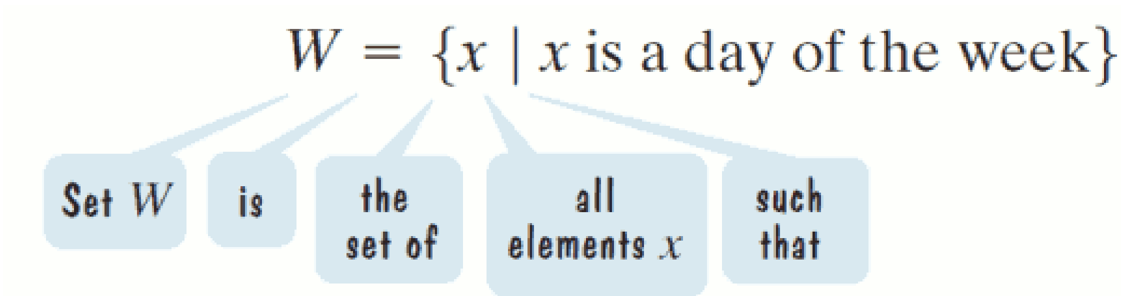
Write using the roster method:

Set *C* is the set of U.S. coins with a value of less than a dollar. Express this set using the roster method.

Solution

C = {penny, nickel, dime, quarter, half-dollar}

Set-Builder Notation



We read this notation as "Set *W* is the set of all elements *x* such that *x* is a day of the week."

Before the vertical line is the variable *x*, which represents an element in general.

After the vertical line is the condition *x* must meet in order to be an element of the set.

Example: Converting from Set-Builder to Roster Notation

Express set

$A = \{x \mid x$ is a month that begins with the letter M$\}$

using the roster method.

Solution

There are two months, namely March and May.

Thus,

$A = \{$ March, May$\}$.

The Empty Set

THE EMPTY SET

The **empty set**, also called the **null set**, is the set that contains no elements. The empty set is represented by { } or $\varnothing$.

These are examples of empty sets:

Set of all numbers less than 4 and greater than 10

$\{x \mid x \text{ is a fawn that speaks}\}$

Example: Recognizing the Empty Set

Which of the following is the empty set?

a. {0}

No. This is a set containing one element.

b. 0

No. This is a number, not a set.

Example: Recognizing the Empty Set

Which of the following is the empty set?

c. {x | x is a number less than 4 or greater than 10}

No. This set contains all numbers that are either less than 4, such as 3, or greater than 10, such as 11.

d. {x | x is a square with three sides}

Yes. There are no squares with three sides.

Notations for Set Membership

THE NOTATIONS $\in$ AND $\notin$

The symbol $\in$ is used to indicate that an object is an element of a set. The symbol $\in$ is used to replace the words "is an element of."

The symbol $\notin$ is used to indicate that an object is *not* an element of a set. The symbol $\notin$ is used to replace the words "is not an element of."

Example: Using the Symbols $\in$ and $\notin$

Determine whether each statement is true or false:

a. $r \in \{a, b, c, \ldots, z\}$

True

b. $7 \notin \{1, 2, 3, 4, 5\}$

True

c. $\{a\} \in \{a, b\}$

False. $\{a\}$ is a set and the set $\{a\}$ is not an element of the set $\{a, b\}$.

Sets of Natural Numbers

THE SET OF NATURAL NUMBERS

$$\mathbf{N} = \{1, 2, 3, 4, 5, \ldots\}$$

The three dots, or ellipsis, after the 5 indicate that there is no final element and that the list goes on forever.

Example: Representing Sets of Natural Numbers

Express each of the following sets using the roster method:

a. Set A is the set of natural numbers less than 5.

$A = \{1,2,3,4\}$

b. Set B is the set of natural numbers greater than or equal to 25.

$B = \{25, 26, 27, 28,\ldots\}$

c. $E = \{ x | x \in \mathbf{N} \text{ and } x \text{ is even}\}$.

$E = \{2, 4, 6, 8,\ldots\}$

Inequality Notation and Sets

Inequality Symbol and Meaning	Example: Set-Builder Notation	Example: Roster Method
$x < a$ — x is less than a.	$\{x \mid x \in \mathbf{N} \text{ and } x < 4\}$ — x is a natural number less than 4.	$\{1, 2, 3\}$
$x \leq a$ — x is less than or equal to a.	$\{x \mid x \in \mathbf{N} \text{ and } x \leq 4\}$ — x is a natural number less than or equal to 4.	$\{1, 2, 3, 4\}$
$x > a$ — x is greater than a.	$\{x \mid x \in \mathbf{N} \text{ and } x > 4\}$ — x is a natural number greater than 4.	$\{5, 6, 7, 8, \ldots\}$
$x \geq a$ — x is greater than or equal to a.	$\{x \mid x \in \mathbf{N} \text{ and } x \geq 4\}$ — x is a natural number greater than or equal to 4.	$\{4, 5, 6, 7, \ldots\}$

Inequality Notation and Sets

$a < x < b$ — x is greater than a and less than b.	$\{x \mid x \in \mathbf{N} \text{ and } 4 < x < 8\}$ — x is a natural number greater than 4 and less than 8.	$\{5, 6, 7\}$
$a \le x \le b$ — x is greater than or equal to a and less than or equal to b.	$\{x \mid x \in \mathbf{N} \text{ and } 4 \le x \le 8\}$ — x is a natural number greater than or equal to 4 and less than or equal to 8.	$\{4, 5, 6, 7, 8\}$
$a \le x < b$ — x is greater than or equal to a and less than b.	$\{x \mid x \in \mathbf{N} \text{ and } 4 \le x < 8\}$ — x is a natural number greater than or equal to 4 and less than 8.	$\{4, 5, 6, 7\}$
$a < x \le b$ — x is greater than a and less than or equal to b.	$\{x \mid x \in \mathbf{N} \text{ and } 4 < x \le 8\}$ — x is a natural number greater than 4 and less than or equal to 8.	$\{5, 6, 7, 8\}$

Example: Representing Sets of Natural Numbers

Express each of the following sets using the roster method:

a. $\{x \mid x \in \mathbf{N} \text{ and } x \leq 100\}$

Solution: {1, 2, 3, 4,…,100}

b. $\{x \mid x \in \mathbf{N} \text{ and } 70 \leq x < 100\}$

Solution: {70, 71, 72, 73, …, 99}

Cardinality and Equivalent Sets

DEFINITION OF A SET'S CARDINAL NUMBER

The **cardinal number** of set A, represented by $n(A)$, is the number of distinct elements in set A. The symbol $n(A)$ is read "n of A."

Repeating elements in a set neither adds new elements to the set nor changes its cardinality.

Example: Cardinality of Sets

Find the cardinal number of each of the following sets:

a. $A = \{ 7, 9, 11, 13 \}$

$n(A) = 4$

b. $B = \{0\}$

$n(B) = 1$

c. $C = \{ 13, 14, 15, \ldots, 22, 23\}$

$n(C)=11$

Equivalent Sets

DEFINITION OF EQUIVALENT SETS

Set A is **equivalent** to set B means that set A and set B contain the same number of elements. For equivalent sets, $n(A) = n(B)$.

Equivalent Sets

These are equivalent sets:

$n(A) = n(B) = 5$

$$A = \{x \mid x \text{ is a vowel}\} = \{a, e, i, o, u\}$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$B = \{x \mid x \in \mathbf{N} \text{ and } 3 \leq x \leq 7\} = \{3, 4, 5, 6, 7\}.$$

The line with arrowheads, $\updownarrow$, indicate that each element of set *A* can be paired with exactly one element of set *B* and each element of set *B* can be paired with exactly one element of set *A*.

Equivalent Sets

ONE-TO-ONE CORRESPONDENCES AND EQUIVALENT SETS

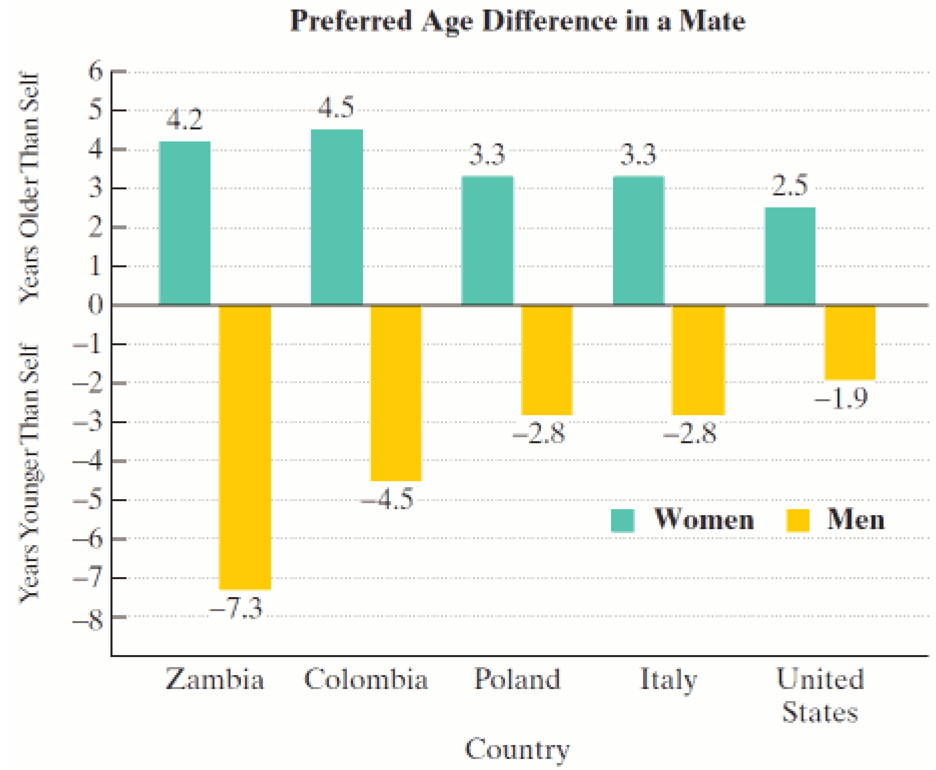
1. If set A and set B can be placed in a one-to-one correspondence, then A is equivalent to B: $n(A) = n(B)$.
2. If set A and set B cannot be placed in a one-to-one correspondence, then A is not equivalent to B: $n(A) \neq n(B)$.

Example: Determining if Sets are Equivalent

This figure shows the preferred age difference in a mate in five selected countries.

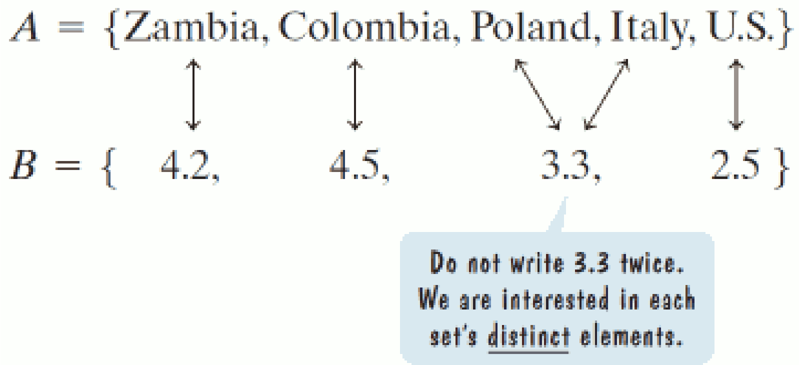
A = the set of five countries shown

B = the set of the average number of years women in each of these countries prefer men who are older than themselves. Are these sets equivalent? Explain.



Example continued

Method 1: Trying to set up a One-to-One Correspondence.

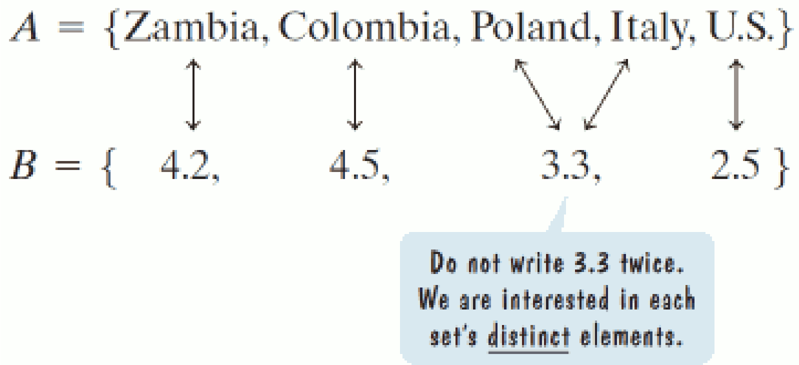


Solution:

The lines with the arrowheads indicate that the correspondence between the sets is not one-to-one. The elements Poland and Italy from set *A* are both paired with the element 3.3 from set *B*. These sets are not equivalent.

Example continued

Method 2: Counting Elements



Solution:

Set A contains five distinct elements: $n(A) = 5$. Set B contains four distinct elements: $n(B) = 4$. Because the sets do not contain the same number of elements, they are not equivalent.

Finite and Infinite Sets

FINITE SETS AND INFINITE SETS

Set A is a **finite set** if $n(A) = 0$ (that is, A is the empty set) or $n(A)$ is a natural number. A set whose cardinality is not 0 or a natural number is called an **infinite set**.

Equal Sets

DEFINITION OF EQUALITY OF SETS

Set A is **equal** to set B means that set A and set B contain exactly the same elements, regardless of order or possible repetition of elements. We symbolize the equality of sets A and B using the statement $A = B$.

Example: Determining Whether Sets are Equal

Determine whether each statement is true or false:

a. {4, 8, 9} = {8, 9, 4}

True

b. {1, 3, 5} = {0, 1, 3, 5}

False